11. WAVES ON A STRING

1. INTRODUCTION

We come across numerous phenomena in nature based on the properties of wave motion. This chapter describes the equations and properties of wave motion. The study of waves on a string forms the basis of understanding the phenomena associated with sound waves and other mechanical and non-mechanical waves. Wave transmits both energy and momentum from one region to other. Mechanical waves require a medium to travel, whereas non-mechanical waves don't. Wave on a string is a mechanical wave but the properties and concepts studied here will be useful in studying non-mechanical waves as well.

2. WAVE MOTION

A wave is a disturbance or variation traveling through space and matter. It is the undulating movement of energy from one point to another. The medium through which the wave passes may experience some oscillations, but the particles in the medium do not travel with the wave. The wave equation, which is a differential equation, expresses the properties of motion in waves. Waves come in all shapes and sizes, and accordingly, the mathematical expression of the wave equation also varies.

2.1 Types of Waves

Waves can, broadly, be classified into two types:

(a) **Mechanical waves**: Waves that require a medium/matter for their propagation are called mechanical waves. These waves are generated due a disturbance in the medium (particles in the matter) and while the wave travels through the medium, the movement of the medium (particles) is minimal. Therefore mechanical waves propagate only energy, not matter. Both the wave and the energy propagate in the same direction. All waves (mechanical or electromagnetic) have a certain energy. Only a medium possessing elasticity and inertia can propagate a mechanical wave.

(b) **Non-mechanical waves/Electromagnetic waves**: Waves that do not require a medium/matter for their propagation are called electromagnetic waves. These waves are formed by the coupling of electric and magnetic fields due to acceleration of electric charge and can travel through vacuum. Depending on the wavelength of the electromagnetic wave, they are classified as radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays, and gamma rays.

3. WAVE PULSE ON A STRING

A wave pulse is a single, sudden, and short-duration disturbance that moves from point A to point B through a medium, e.g., a string. We know that waves originate when a disturbance at the source point moves through one particle to its adjacent particles from one end of the medium to the other. Now, when a disturbance-producing source active for a short time, a wave pulse passes through the medium. Conversely, when the source remains active for an extended time, creating a series of motions, it results in a wave train or a wave packet. Thus, a wave train is a group of waves traveling in the same direction.

For example, if the person in figure decides to move his hand up and down 10 times and then stop, a wave train consisting of 10 loops will move on the string.

4. EQUATION OF A TRAVELING WAVE

In the figure, let us assume that the man starts moving his hand at t = 0 and finished his job at t = Δt. The vertical displacement of the left end of the string, denoted as y, is a function of time. It is zero for $t<0$ and $t>\Delta t$. This function can be represented by f (t). Let us take the left end of the string as the source of the wave and take the X axis along the string toward right. The function f (t) represents the displacement y of the particle at x=0 as a function of time: y(x=0, t) =f (t).

The disturbance on the string travels towards right at a constant speed. Thus, the displacement produced at the left end at time t reaches the point x at time $t+\left(\frac{x}{v}\right)$ Similarly, the displacement of the particle at point x at time t was generated at the left end at the time t-x/. But the displacement of the left end at time t-x/v is f (t-x/v). Hence, y (x, t) = y(x=0, t-x/v) = f(t-x/v).

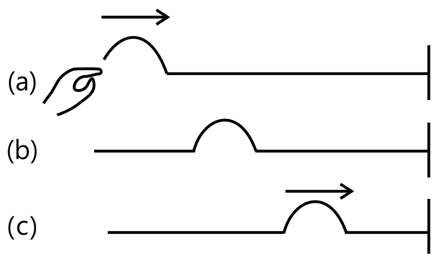


Figure 11.1

The displacement of the particle at x at time t, i.e., y(x, t) is generally abbreviated as y and the wave equation is written as y = f (t-x/v). ... (i)

Equation (i) represents a wave traveling in the positive direction x at a constant speed. Such a wave is called a traveling wave or a progressive wave. The function f is dependent on the movement of the source, and therefore, arbitrary. The time t and the position x must be represented in the wave equation in the form t-x/ only. For example,

$y=A\sin\frac{(t-x/v)}{T}$, and $y=Ae^{-\frac{(t-x/v)}{T}}$ are valid wave equations.

Both these equations represent the movement of the wave in the positive direction x at constant speed v.

In contrast, the equation $y=A\sin\frac{(x^2-v^2t^2)}{L^2}$ does not represent the movement of the wave in the direction x at a constant speed . If a wave travels in the negative direction at a speed , its general equation may be written as

y = f(t + x/v) ... (ii)

Equation (i) can also be written as $y=f\left(\frac{vt-x}{v}\right)$ or $y=g(x-vt)$, ...(iii)

where g is some other function having the following meaning: Let us assume that t = 0 in the wave equation. Then, we get the displacement of various particle at t = 0, i.e., $y=(x,t=0)=g(x)$. Thus, the function g(x) represents the shape of the string at t = 0. Assuming that the displacement of the different particles at t = 0 is represented by the function g(x), the displacement of the particle at x at time t will be y = g(x - vt). Similarly, if the wave is traveling along the negative direction x and the displacement of a different particle at t = 0 is g(x), the displacement of the particle at x at time t will be y = g(x + t) ...(iv)

Illustration 1: The wave equation of a wave propagating on a stretched string along its length taken as the positive x axis is given as $y=y_0\exp\left(-\left(\frac{t}{T}-\frac{x}{\lambda}\right)^2\right)$ where y_0 = 4 mm, T = 1.0 s and λ = 4 cm.

(a) Find the velocity of the wave.

(b) Find the function f(t) giving the displacement of the particle at x = 0.

(c) Find the function g(x) giving the shape of the string at t = 0.

(d) Plot the shape g(x) of the sting at t = 0.

(e) Plot the shape of the string at t = 5s. **(JEE MAIN)**

Sol: The wave moves having natural frequency of ν and wavelength λ has velocity $V = \nu\lambda$.

As the frequency is $\nu = \frac{1}{T}$ the velocity of the wave is then $V = \frac{\lambda}{T}$.

(a) The wave equation can be written as $y = y_0 e^{-\frac{1}{T^2}\left(t - \frac{x}{\lambda/T}\right)^2}$

On comparison with the general equation y = f (t – x/), we can infer that, $\upsilon = \frac{\lambda}{T} = \frac{4\,cm}{1.0\,s} = 4\,cms^{-1}$

(b) Putting x =0 in the given equation $f(t) = y_0\, e^{-(t/T)^2}$ ... (i)

(c) Putting t = 0 in the given equation $g(t) = y_0\, e^{-(x/\lambda)^2}$ ... (ii)

(d)

(e)

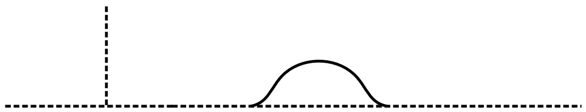


Figure 11.2

4.1 Sine Wave Traveling on a String

Consider the scenario where the person in the Fig. 11.3 keeps moving his hand up and down continuously. As energy is being constantly supplied by the person, the wave generated at the source keeps oscillating the any part of the string through which it passes. Thus energy passes from the left (the source) to the right continuously till the person gets tired. The nature of the vibration of any particle in the string is similar to that of the left end (the source), the only difference is that there is an interval of x/ between two motions.

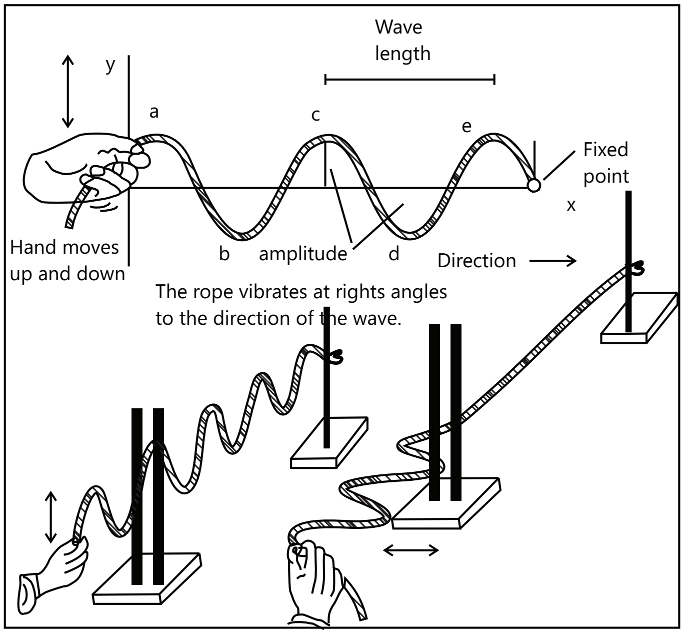


Figure 11.3

When the person in the Fig 11.3 oscillates the left end x = 0 in a simple harmonic motion, the equation of motion of this end may be written as f(t) = Asin ω t ... (i)

where A is the amplitude and ω is the angular frequency. The time period of oscillation is given by $T=2\pi/\omega$ and the frequency of oscillation is

$v = 1/T = \omega/2\pi$. The wave produced by such an oscillation source is called a sine wave or sinusoidal wave.

The displacement of the particle at x at time t will be

$$y = f(t - x/\upsilon) \quad \text{or} \quad y = A\sin w(t - x/\upsilon) \qquad \text{... (ii)}$$

The velocity of the particle at x at time t is given by $\frac{\partial y}{\partial t} = A\omega\cos(t - x/\upsilon)$ ... (iii)

MASTERJEE CONCEPTS

- While differentiating with respect to t, we should treat x as constant – it is the same particle whose displacement should be considered as a function of time. Therefore, the symbol $\frac{\partial}{\partial t}$ is used in place of $\frac{d}{dt}$.
- In the event that the waves travel along negative x direction, the direction of V_p will change.

  Particle velocity is the same as wave velocity. The two are totally different. While the wave moves on the string at a constant velocity along the x axis, the particle moves up and down with velocity $\frac{\partial}{\partial t}y$, which changes with x and t.

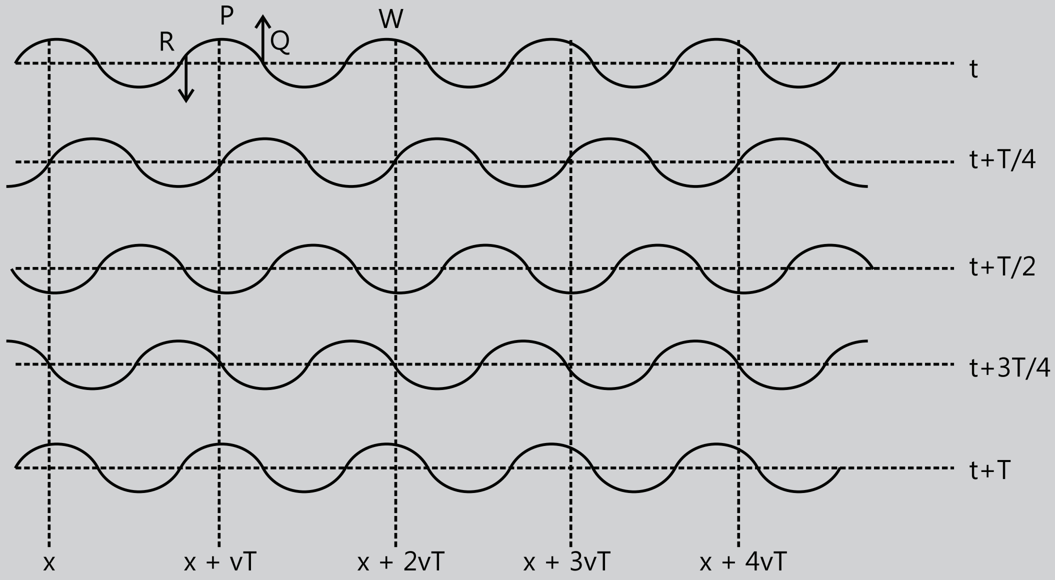


Figure 11.4

Above figure shows change in shape of string with time

Vaibhav Krishan (JEE 2009 AIR 22)

4.1.1 Some Important Terms

(a) Amplitude: In a wave, the crest represents highest point the wave rises to and equilibrium represents the default position from which a wave arises. Therefore, the distance between the crest and the equilibrium point in a single wave cycle is referred to as the equilibrium.

(b) Wavelength: The distance between any two points with the same phase, such as between crests or troughs is referred to as the wavelength λ. It is generally measured in meters.

(c) Wave Number: Wave number is a measurement of a certain number of wavelengths for some given distance. In a sense, the wave number is like a spatial analogue of frequency. Typically, wave number is taken to be 2π times the number of wavelengths per unit of distance, which is the number of radians for each unit of distance as well. $k = \frac{2\pi}{\lambda}$

(d) Time Period: A period T is the time needed for one complete cycle of vibration of a wave to pass a given point.

(e) Frequency: Frequency describes the number of waves that pass a fixed place in a given amount of time and is typically measured in hertz. These are related by $f = \frac{1}{T}$

(f) Angular Frequency: The angular frequency ω gives the frequency with which phase changes. It is expressed in radians per second. It is related to the frequency or period by $\omega = 2\pi f = \frac{2\pi}{T}$ ....(i)

Illustration 2: Consider the wave y = (5 mm) sin [(1 cm $^{-1}$) x – (60s $^{-1}$) t]. Find (a) the amplitude, (b) the wave number, (c) the wavelength, (d) the frequency, (e) the time period and (f) the wave velocity. **(JEE MAIN)**

Sol: Comparing the given equation with $y = A\sin(kx - \omega t)$ we get the values of wave number k, amplitude A and angular frequency. The frequency $\omega = 2\pi\upsilon = 2\pi/T$. The velocity of wave is $v = \nu\lambda$ and the wave number of wave is $K = \frac{2\pi}{\lambda}$.

On comparing the given equation with standard equation of a traveling wave, we find

(a) Amplitude A = 5mm, (b) wave number k = 1 cm^{-1}, (c) wavelength $\lambda = \frac{2\pi}{k} = 2\pi\,\text{cm}$

(d) Frequency $\nu = \frac{\omega}{2\pi} = \frac{60}{2\pi}\text{Hz} = \frac{30}{\pi}\text{Hz}$ (e) Time period $T = \frac{1}{\nu} = \frac{\pi}{30}\text{s}$

(f) Wave velocity $v = \nu\lambda = 60\ \text{cms}^{-1}$

4.2 Velocity of Waves on a String

The wave speed depends on the properties of the medium. For a string, the speed of a transverse wave traveling along a vibrating string (v) is directly proportional to the square root of the tension of the string (T) over the linear mass density (μ):

$v = \sqrt{\frac{T}{\mu}}$, where the linear density μ is the mass per unit length of the string ....(i)

4.3 Phase Difference

The amount by which two cyclical motions of the same frequency, are out of step with each other. It can be measured in degrees from 0° to 360°, radians from 0 to 2π, or seconds of time.. If two oscillators have the same frequency and no phase difference, they are said to be in phase. Conversely, if they have the same frequency and different phases, then they have a phase difference and they are said to be out of phase with each other. If the phase difference is 180° (π radians), then the two oscillators are said to be in anti-phase.

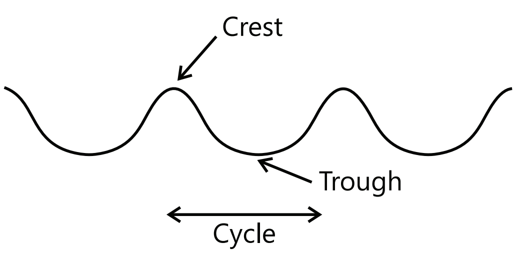


Figure 11.5

4.4 Crest and Trough

In a wave, the crest represents highest point the wave rises within a cycle. A trough is the opposite of a crest, hence the minimum or lowest point in a cycle.

5. ALTERNATIVE FORMS OF WAVE EQUATION

As seen earlier, the wave equation of a wave traveling in x direction is $y = A\sin\omega(t - x/\upsilon)$,

This can also be written in several other forms such as $y = A\sin(\omega t - kx)$, ... (i)

$$y = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad \text{... (ii)}$$

$$y = A\sin[k\ (vt - x)] \quad \text{... (iii)}$$

Please bear in mind our choice of t = 0 in writing equation (v) from which the wave equation has been derived. Also, the point at which the left end x = 0 crosses its mean position y = 0 and goes up has been chosen as the origin of time. For a general choice of the origin of time, a phase constant will have to be added to give the equation

$$y = A\sin[\omega(t - x/v) + \phi] \quad \text{... (iv)}$$

The constant φ will be $\pi/2$ If we choose t = 0 at an instant when the left end reaches its extreme position y=A, then the constant φ will be $\pi/2$. The equation will then be

$$y = A\cos\omega(t - x/v), \quad \text{... (v)}$$

If on the other hand, t = 0 is taken at the point when the left end is crossing the mean position from an upward to downward direction, φ will be π and the equation will be

$$y = A\sin\omega\left(\frac{x}{v} - t\right) \text{ or } y = A\sin(kx - \omega t) \quad \text{... (vi)}$$

MASTERJEE CONCEPTS

Both $\sin(kx - \omega t)$ and $\sin(\omega t - kx)$ differ just by a phase of "π". If a particle at t = 0, x = 0 in its mean position is moving upwards (in first wave), then the same particle would be in mean position and the particle would be moving down!

B Rajiv Reddy (JEE 2012, AIR 11)

Illustration 3: Fig 11.6 shows a string of linear mass density 1.0 Kg m^{-1} and a length of 50 cm. Find the time taken by a wave pulse to travel through the length of the string. Take g = 10 ms^{-2}. **(JEE MAIN)**

Sol: The wave velocity on stretched string under tension F = mg is given by

$v = \sqrt{\frac{F}{\mu}}$ where μ is mass per unit length of the string.

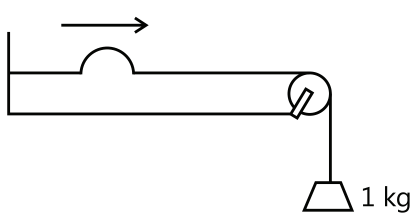


Figure 11.6

The tension in the string is F = mg = 10N. Given that the mass per unit length is μ = 1.0 Kg m^{-1}, the wave velocity is, $v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{10N}{0.1kgm^{-1}}} = 10ms^{-1}$.

Therefore, to travel through 50 cm, the wave pulse will take 0.05 s.

Illustration 4: A rubber tube that is 12.0 m long and that has a total mass of 0.9 kg is fastened to fixed base. At the other end of the tube, a cord is attached that passes over a pulley and supports an object with a mass of 5.0 kg. If the tube is struck at one end, find the time required for the transverse pulse to reach the other end. ($g = 9.8\ m/s^2$) **(JEE MAIN)**

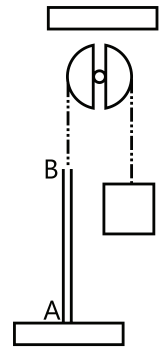


Figure 11.7

Sol: For the string under the tension T =mg where m is mass of the block. When the rod is struck at lower rod, the wave thus originated travels at speed $v=\sqrt{\frac{T}{\mu}}$ where μ is the mass per unit length of the string.

Tension in the rubber tube AB, T = mg or T = (5.0) (9.8) = 49 N

Mass per unit length of rubber tube, $\mu=\frac{0.9}{12}=0.075\,kg/m$

$\therefore$ Speed of wave on the tube, $v=\sqrt{\frac{T}{\mu}}=\sqrt{\frac{49}{0.075}}=25.56\,m/s$

$\therefore$ The required time is $t=\frac{AB}{v}=\frac{12}{25.56}=0.47\,s$.

Illustration 5: Prove that the equation y = a sin (ωt – kx) satisfies the wave equation $\frac{\partial^2 y}{\partial t^2}=v^2\frac{\partial^2 y}{\partial x^2}$ and find speed of the wave and the direction in which it is traveling. **(JEE ADVANCED)**

Sol: To prove the above relation, we need to take the ratio of second order time derivative of wave equation and second order displacement derivative of wave equation.

$\frac{\partial^2 y}{\partial t^2}=-\omega^2 a\sin(\omega t-kx)$ and $\frac{\partial^2 y}{\partial x^2}=k^2 a\sin(\omega t-kx)$. We can write these two equation as,

$\frac{\partial^2 y}{\partial t^2}=\frac{\omega^2}{k^2}.\frac{\partial^2 y}{\partial x^2}$. Comparing this with, $\frac{\partial^2 y}{\partial t^2}=v^2\frac{\partial^2 y}{\partial x^2}$

We get, wave speed $v=\frac{\omega}{k}$

The negative sign between ωt and kx implies that wave is traveling along positive direction.

Illustration 6: The Fig 11.8 shows a snapshot of a sinusoidal traveling wave which was taken at t = 0.3 s. The wavelength is 7.5 cm and the amplitude is 2 cm. Assuming the crest was at x = 0 at t = 0, write the equation of traveling wave. **(JEE ADVANCED)**

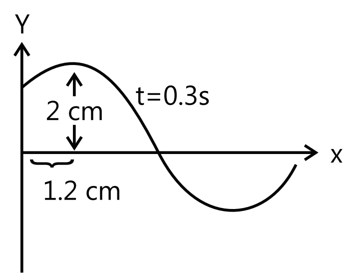


Figure 11.8

Sol: The equation of travelling wave is $y=A\sin(kx-\omega t)$. The wave number is given by $k=\frac{2\pi}{\lambda}$ and angular frequency of wave is $\omega=vk$.

Given, A = 2 cm, $\lambda=7.5$ cm $\therefore$ $k=\frac{2\pi}{\lambda}=0.84\,cm^{-1}$

The wave has traveled a distance of 1.2 cm in 0.3 s. Hence, speed of the wave $v=\frac{1.2}{0.3}=4\,cm/s$

$\therefore$ Angular frequency ω = (v) (k) = 3.36 rad/s

Since the wave is traveling along the positive direction x, and crest (maximum displacement) is at x = 0 at t = 0, we can write the wave equation as

Y(x, t) = A cos (kx - ωt) or y(x, t) = A cos (ωt - kx) as cos(-θ) = cos θ

Therefore, the equation of the traveling wave is

y(x, t) = (2 cm) cos [(0.84 cm-1)x – (3.36 rad/s)t]

Illustration 7: The mass and length of a rope hanging from the ceiling are 0.1 kg and 2.45 m, respectively. The rope has a uniform width.

(a) Determine the speed of transverse wave in the rope at a point 0.5 m away from the lower end.

(b) Also, calculate the time taken by the wave to travel the full length of the rope.

(JEE ADVANCED)

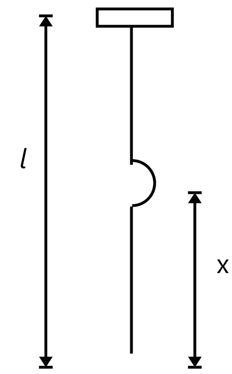


Figure 11.9

Sol: As the rope hangs under it's own weight, the tension in string at a distance x from hanging end is $T = mg\frac{x}{\ell}$ where ℓ is the length of the string and m is mass of the string.

When a transverse waves are generated to travel along length of rope, they travel with speed $v = \sqrt{\frac{T}{\mu}}$ where μ is mass per unit length of string.

The tension in the string will be different at different points owing to the mass of the string and the fact that it is suspended vertically from a ceiling.. The tension at a point which is at a distance x free end will be due to the weight of the string below it. Given that m is the mass of string of length l, the mass of length x of the string will be $\left(\frac{x}{\ell}\right)m$

$$\mu = \frac{0.1}{2.45} = 0.04\text{kg/m; Tension} = mg\left(\frac{x}{\ell}\right) = mg\left(\frac{0.5}{2.45}\right) = 0.20\text{N} \Rightarrow v = \sqrt{\frac{T}{\mu}} = 2.236\text{m/s}$$

(b) From the above equation, we see that velocity of the wave is different at different points. Therefore, if at point x the wave travels a distance dx in time dt, then $dt = \frac{dx}{v} = \frac{dx}{\sqrt{gx}}$

$$\therefore \qquad \int_0^1 dt = \int_0^1 \frac{dx}{\sqrt{gx}}; \qquad t = 2\sqrt{\frac{\ell}{g}} = 2\sqrt{\frac{2.45}{9.8}} = 1.0\text{s}$$

Illustration 8: The mass and length of a rope hanging vertically from a rigid support are 12 m and 6 kg, respectively. A stone of mass 2 kg is attached to the free end of the rope. The rope has a uniform width. If a transverse pulse of wavelength 0.06 m is produced at the lower end of the rope, what will be the wavelength of the pulse when it reaches the top of the rope?

(JEE ADVANCED)

Sol: The wave velocity will be $V = \nu\lambda = \sqrt{\frac{F}{\mu}}$ where F is the tension in rope at a point and μ is mass per unit length of the string. As F is varying along the length of the rope so the velocity will vary along the length of the rope. As source frequency is constant λ will vary.

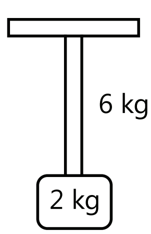


Figure 11.10

Owing to the fact that a stone is attached to the lower end of the rope, the tension in the rope will be different at the different points. The tension at the lower end will be 20 N and at the upper end it will be 80 N.

We have, $V = \nu\lambda$ or, $\sqrt{\frac{F}{\mu}} = \nu\lambda$ or, $\frac{\sqrt{F}}{\lambda} = \nu\sqrt{\mu}.$

The frequency of the wave pulse is affected only by the frequency of the source, and hence the wave pulse frequency will be the same across the length of the rope as it depends only on the frequency of the source. As the rope has a uniform width, the mass per unit length will also be consistent across the length of the rope.

Thus, by (i) $\frac{\sqrt{F}}{\lambda}$ is constant.

Hence, $\frac{\sqrt{(2kg)g}}{0.06m} = \frac{\sqrt{(8kg)g}}{\lambda_1}$ where λ_1 is the wavelength at the top of the rope. This gives λ_1 = 0.12m

6. POWER TRANSMITTED ALONG THE STRING BY A SINE WAVE

The direction of a traveling wave on a string and the direction of the energy transmitted by it is the same. Consider a sine wave traveling along a stretched string in the direction x. The equation for the displacement in the direction y is $y = A \sin \omega (t - x/v)$ ... (i)

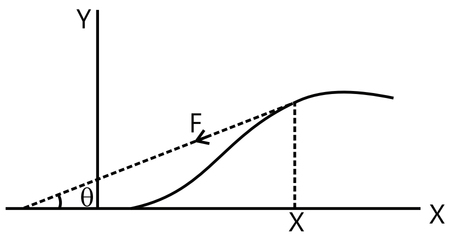


Figure 11.11

The Fig 11.11 the portion of the string to the left of the point x exerts a force F on the portion of the string to the right of the point x at time t. The direction of this force is along the tangent to the string at position x. The component of the force along the axis y is $F_y = -F\sin\theta \approx -F\tan\theta = -F\frac{\partial y}{\partial x}$

The power delivered by the force F to the string on the right of position x is, therefore, $P = \left(-F\frac{\partial y}{\partial x}\right)\frac{\partial y}{\partial t}$

By (i), it is $-F\left[\left(-\frac{\omega}{v}\right)A\cos\omega(t-x/v)\right][\omega A\cos\omega(t-x/v)] = \frac{\omega^2 A^2 F}{v}\cos^2\omega(t-x/v)$

This is the rate at which energy is being transferred from left portion of the string to the right portion across the point at x. The $\cos^2$ term oscillates between 0 and 1 during cycle and its average value is 1/2, therefore, the average power transmitted across any point is

$$P_{av} = \frac{1}{2}\frac{\omega^2 A^2 F}{v} = 2\pi^2 \mu c A^2 v^2 \qquad \text{... (ii)}$$

The power passing along the length of the string is proportional to the square of the amplitude and square of the frequency of the wave.

Illustration 9: For a sine wave with an amplitude of 2.0 mm, the average power transmitted through a given point on a string is 0.20 W. What will be the power that will be transmitted through this point were the amplitude to be increased to 3.0 mm?. **(JEE ADVANCED)**

Sol: The power transmitted by the sine wave is $P \propto A^2$ where A is the amplitude of the wave.

Other things being equal, the power transmitted is proportional to the square of the amplitude.

Thus, $\frac{P_2}{P_1} = \frac{A_2^2}{A_1^2}$ or $\frac{P_2}{0.20W} = \frac{9}{4} = 2.25$ $P_2 = 2.25 \times 0.20\ W = 0.45\ W$

7. ENERGY IN WAVE MOTION

Every wave motion involves transfer of energy and momentum.. Waves are produced when force is applied to a portion of the wave medium. When force is applied to a portion of the wave medium, the disturbance thus caused in that portion of the medium generates a wave that exerts a force on the adjoining portions. This, in turn, disturbs those portions, thereby propagating the wave further to the adjacent portions. In this way, a wave can transport energy from one region of space to other.

The energy in wave motion is manifested in three forms, namely, energy density (u), power (P), and intensity (I). We shall discuss them one by one.

7.1 Energy Density (μ)

The energy density of a progressive wave is the total mechanical energy (kinetic + potential) per unit volume of the medium through which the wave is propagated. This can be illustrated through an example. Let us imagine a string attached to a tuning fork. When the tuning fork is struck, the vibration transmits energy to the segment of the string attached to it, or in other words, as the vibrating fork moves through its equilibrium position, it stretches a segment of the string, increasing its potential energy, while also imparting transverse speed to the segment, increasing its kinetic energy. Thus, as the wave moves along the string, energy is transferred to the other segments of the string.

7.2 Kinetic Energy Per Unit Volume

The kinetic energy of a unit volume of the string can be calculated from the wave function. Mass of unit volume is the density ρ. Its displacement from equilibrium is the wave function

$y = A \sin(kx - \omega t)$.

Its speed is $\frac{dy}{dt}$, where x is considered to be fixed. The kinetic energy of unit volume ΔK is then

$$\Delta K = \frac{1}{2}(\Delta m)v_y^2 = \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2; \quad \text{Using } y = A\sin(kx - wt)\text{, we obtain } \frac{dy}{dt} = -wA\cos(kx - wt)$$

So the kinetic energy per unit volume is $\Delta K = \frac{1}{2}\rho^2\omega^2A^2\cos^2(kx - \omega t)$ ... (i)

7.3 Potential Energy Per Unit Volume

The work done by the vibrating fork by stretching the segment of the string is the potential energy of the segment. It depends on the slope $\frac{dy}{dx}$. The potential energy per unit volume of the string is related to the slope and tension T and is given by (for small slopes)

$$\Delta U = \frac{1}{2}\rho v^2\left(\frac{dy}{dx}\right)^2 \quad \text{... (ii)}$$

where $\qquad v = \text{wave speed} = \frac{\omega}{k}$

Using $\frac{dy}{dx} = kA\cos(kx - \omega t)$, we obtain for the potential energy $\Delta U = \frac{1}{2}\rho\omega^2A^2\cos^2(kx - \omega t)$ ... (iii)

which is the same as the kinetic energy. The total energy per unit volume is $\Delta E = \Delta K + \Delta U = \rho\omega^2A^2\cos^2(kx - \omega t)$ ... (iv)

The total energy per unit volume (ΔE) varies with time. As the average value of $\cos^2(kx - \omega t)$ at any point is $\frac{1}{2}$, the average energy per unit volume (also called the energy density μ) is $\mu = \frac{1}{2}\rho\omega^2A^2$ ... (v)

MASTERJEE CONCEPTS

In the case of a spring with mass ρ attached to it and oscillating in a simple harmonic wave, the energy density is the same as in equation (v). However, its potential energy is maximum when the displacement is maximum. In the case of a string segment, it is the slope of the spring that determines the potential energy and it is maximum when the slope is maximum, which is at the equilibrium position of the segment – the same position for which the kinetic energy is maximum.

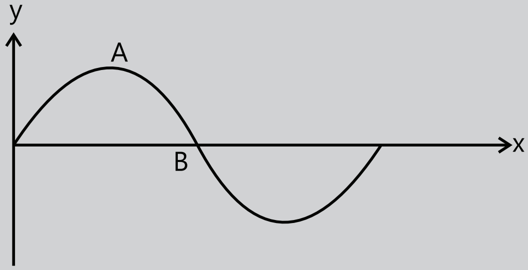


Figure 11.12

In the Fig 11.12, the kinetic energy and potential energy both are zero at point A, whereas at point B, both the kinetic energy and potential energy are maximum.

Aman Gour (JEE 2012, AIR 230)

7.4 Intensity (I)

The intensity of a wave is defined as the flow of energy per unit area of a cross-section of the string in unit time.

Thus, $I = \dfrac{\text{power}}{\text{area of cross} - \text{section}} = \dfrac{P}{s}$ or $I = \dfrac{1}{2}\rho\omega^2 A^2 V$

This is, however, the average intensity transmitted through the string. The instantaneous intensity $\rho\omega^2 A^2 v \sin^2(kx - \omega t)$ or $\rho\omega^2 A^2 v\cos^2(kx - \omega t)$ depends on x and t.

MASTERJEE CONCEPTS

- The relation for power and intensity discussed above are for transverse waves on a string. However, they hold good for other waves also.
- Intensity due to a point source: Assuming that waves are propagated uniformly in all directions, the energy at a distance r from a point source is distributed uniformly on a spherical surface of radius r and area $S = 4\pi r^2$. If P is the power per unit area that is incident perpendicular to the direction of propagation, then intensity $I = \dfrac{P}{4\pi r^2}$ or $I \propto \dfrac{P}{r^2}$

  Since amplitude $A \propto \sqrt{I}$, a spherical harmonic wave emanating from a point source can therefore, be written as $y(r,t) = \dfrac{A}{r}\sin(kr - \omega t)$

T P Varun (JEE 2012, AIR 64)

Illustration 10: An oscillator attached to stretched string with a diameter of 4 mm transmits transverse waves through the length of the string. The amplitude and frequency of the oscillation are 10^{-4} m and 10 Hz, respectively. Tension in the string is 100 N, mass density of wire is 4.2×10^3 kg/m^3.

Find: (a) The wave equation along the string

(b) The energy per unit volume of the wave

(c) The average energy flow per unit time across any section of the string

(d) The power required to drive the oscillator. **(JEE ADVANCED)**

Sol: The wave equation of string is $y = A\sin(kx - \omega t)$ where the wave number $k = \frac{2\pi}{\lambda}$, the angular frequency $\omega = 2\pi\nu = \frac{2\pi}{T}$. λ is the wavelength and T is the time period of wave. As the string is under tension of 100 N, the wave velocity on string is given by $V = \nu\lambda = \sqrt{\frac{T}{\mu}}$. Use the formula for wave energy in the string .

(a) Speed of transverse wave on the string is, $V = \sqrt{\frac{T}{\rho S}}$ $(\because \mu = \rho S)$

Substituting the values, we have

$$v = \sqrt{\frac{100}{(4.2\times10^3)\left(\frac{\pi}{4}\right)(4.0\times10^{-3})^2}} = 43.53\ m/s;\ \omega = 2\pi f = 20\pi\frac{rad}{s} = 62.83\frac{rad}{s}$$

Wave number is $k = \frac{\omega}{V} = 1.44\ m^{-1}$

$\therefore$ The wave equation is $y(x,t) = A\sin(kx - \omega t) = (10^{-4}m)\sin\left[(1.44m^{-1})x - \left(62.83\frac{rad}{s}\right)t\right]$

(b) Energy per unit volume of the string, u = energy density = $\frac{1}{2}\rho\,\omega^2 A^2$

Substituting the values, we have $u = \left(\frac{1}{2}\right)(4.2\times10^3)(62.83)^2(10^{-4})^2 = 8.29\times10^{-2}\ J/m^3$

(c) Average energy flow per unit time, $P =$ power $= \left(\frac{1}{2}\rho\omega^2 A^2\right)(sv) = (u)(sv)$

Substituting the values, we have $P = (8.29\times10^{-2})\left(\frac{\pi}{4}\right)(4.0\times10^{-3})^2(43.53) = 4.53\times10^{-5}\ J/s$

(d) Power required to drive the oscillator is obviously 4.53×10^{-5} W.

8. INTERFERENCE

Interference is a phenomenon that occurs when two waves superimpose while traveling in the same medium. This results in the formation of a wave of greater or lower amplitude. Interference happens with waves that emerge from the same source or have the similar frequencies.

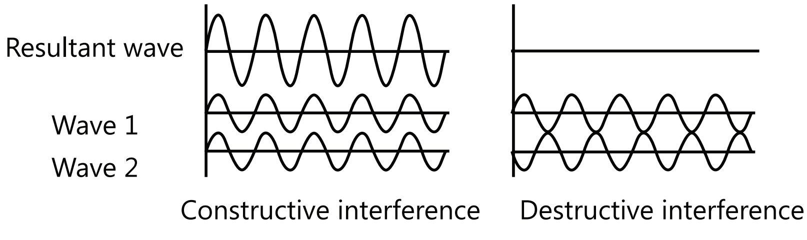


Figure 11.13

8.1 Principle of Superposition

The principle of superposition of waves states that when two or more waves of same type come together at a single point in space, the total displacement at that point is equal to the sum of the displacements of the individual waves. Constructive interference is the meeting of two waves of equal frequency and phase, i.e., if the crest of a wave meets a crest of another wave of the same frequency at the same point, then the total displacement is the sum of the individual displacements. Destructive interference is the meeting of two waves of equal frequency and opposite phase, i.e., if the crest of one wave meets a trough of another wave then the total displacement is equal to the difference in the individual displacements.

In constructive interference, the phase difference between the waves is a multiple of 2π, whereas in a destructive interference the difference is an odd multiple of π. If the phase difference is between these two extremes, then the total displacement of the summed waves lies between the minimum and maximum values. If the first wave alone were traveling, the displacement of particles may be written as $y_1=f_1(t - x / v)$. If the second wave alone were traveling, the displacement may be written as $y_2=f_2(t + x / v)$

If both the waves are traveling on the string, the displacement of its different particles will be given by

$y = y1 + y2 = f_1(t - x / v) + f_2(t + x / v)$.

If the two individual displacements are in opposite directions, the magnitude of the resulting displacement may be smaller than the magnitudes of the individual displacements. In a nutshell, when two or more waves pass through a point at the same time, the disturbance at the point is the sum of the disturbances each wave would produce in absence of the other wave(s).

8.2 Interference of Wave Going in Same Direction

Let us assume that two identical sources send sinusoidal waves of same angular frequency ω in the positive direction x. It is also assumed that the wave velocity and consequentially, the wave number k is same for the two waves. One source may send the wave a little later than the other or the two sources may be located at different points. Here, the phases of the two waves at the point of interference will be different. If we assume the amplitudes of the two waves to be A_1 and A_2 and the phase difference of the two waves to be an angle δ, their equations may be written as

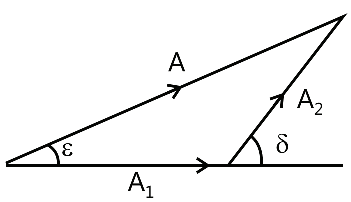


Figure 11.14

$y_1 = A_1 \sin(kx - \omega t)$ And $y_2 = A_2 \sin(kx - \omega t + \delta)$

According to the principle of superposition, the resultant wave is represented by

$y = y_1 + y_2 = A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t + \delta)$

$= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t) \cos\delta + A_2 \cos(kx - \omega t) \sin\delta$

$= \sin(kx - \omega t)(A_1 + A_2 \cos\delta) + \cos(kx - \omega t)(A_2 \sin\delta)$

We can evaluate it using the method to combine two simple harmonic motions. If we write

$A_1 + A_2 \cos\delta = A \cos \varepsilon$ ... (i)

And $A_2 \sin\delta = A \sin \varepsilon$ ... (ii)

We get, $y = A[\sin(kx - \omega t)\cos\varepsilon + \cos(kx - \omega t)\sin\varepsilon] = A\sin(kx - \omega t + \varepsilon)$

Thus, the resultant is indeed a sine wave of amplitude A with a phase difference ε with the first wave. By (i) and (ii),

$A^2 = A^2\cos^2\varepsilon + A^2\sin^2 \varepsilon = \left(A_2 + A_2\cos\delta\right)^2 + \left(A_2\sin\delta\right)^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos\delta$

Or $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos\delta}$ ... (iii)

Also $\tan\varepsilon = \dfrac{A\sin\varepsilon}{A\cos\varepsilon} = \dfrac{A_2 \sin\delta}{A_1 + A_2 \cos\delta}$ ... (iv)

These relations may be remembered by using the following geometrical model can be used to remember these relations: draw a vector of length A_1 to represent $y_1 = A_1\sin(kx - \omega\tau)$ and another vector of length A_2 at an angle δ with the first one to represent $y_2 = A_2 \sin(kx - \omega\tau + \delta)$. The resultant vector then represents the resultant wave $y=A \sin(kx - \omega\tau + \varepsilon)$. The given Fig 11.14 shows the construction.

Illustration 11: The equations of two waves passing simultaneously through a string are given by $y_1 = A_1\sin k(x - vt)$ and $y_2 = A_2 \sin k(x - vt + x_0)$, where the wave number $k = 6.28\ cm^{-1}$ and $x_0 = 1.50$ cm. The amplitudes for A_1 and A_2 are 5.0 mm and 4.0 mm, respectively. Find the phase difference between the waves and the amplitude of the resulting wave. **(JEE ADVANCED)**

Sol: As there are two waves passing through the string simultaneously, the phase difference between the two waves will be $\delta = kx_0$. And the resulting amplitude of the waves will be $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$.

The phase of the first wave is k (x -vt) and of the second is k (x- vt + x_0).

The phase difference is, therefore, $\delta = kx_0 = (6.28\text{cm}^{-1})\ (1.50\text{ cm}) = 2\pi \times 1.5 = 3\pi$

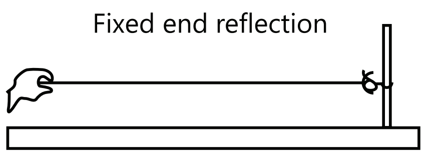
We can thus infer that this is a destructive interference. The amplitude of the resulting wave is given by $|A_1 - A_2|$ = (5.0 − 4.0) mm = 1.0mm.

9. BOUNDARY BEHAVIOUR

When a propagating wave reaches the end of the medium it encounters an obstacle or, maybe, another medium through which it could travel. Here, the interface of the two media is referred to as the boundary and the behavior of a wave/pulse at that boundary is described as its boundary behavior.

9.1 Fixed End Reflection

Let us consider an elastic string which is attached at one end to a pole on a lab bench while the other end is will be held in the hand and stretched in order to introduce pulses into the medium. The end of the elastic string that is attached to the pole is immovable when a wave or pulse reaches it. If a pulse is introduced at the hand-held end of the rope, it will travel through the string towards the fixed immovable end of the medium. This is called the incident pulse since it is incident (i.e., approaching toward) the boundary with the pole. With the incident pulse reaches the boundary, two things occur:



An elastic rope security to a can be used to study the behavior waves at a fixed end.

Figure 11.15

(a) Some of the energy transmitted by the pulse is reflected back towards the hand-held end of the rope. This is known as the reflected pulse.

(b) That part of the energy that is transmitted to the pole causes the pole to vibrate.

As the vibrations of the pole are not obvious, the energy transmitted to it is not typically discussed. The emphasis here will be on the reflected pulse. What are the characteristics/properties of its motion?

When seen from the fixed immovable end, the reflected pulse is a mirror image of incident pulse. That is, an upward displaced pulse will be reflected and returned as a downward displacement pulse and vice-versa.

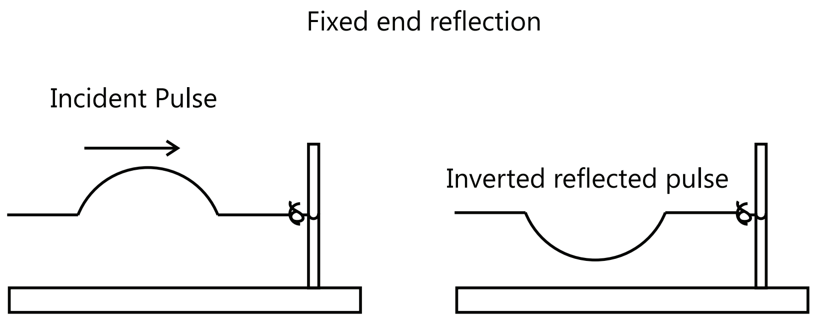
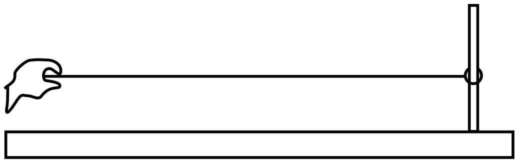


Figure 11.16

9.2 Free End Reflection

Continuing with the above example, let us consider the situation where instead of being securely attached to a lab pole, the elastic string is attached to a ring that is fixed loosely around the pole. Since the string is no longer attached firmly to the pole, the last particle of the rope will be able to move when a pulse reaches it.

Now, if a pulse is introduced at the hand-held end of the string, it will travel through the string towards the pole at the right end of the medium. However, the string is no longer fixed tightly to



if the end of an elastic roe not fastened to the pole then it will be free move up and down. This provides for the study of waves behavior at free ends.

Figure 11.17

the pole and, therefore, the string and the pole will slide past each other. There will be no interaction between the string particle and the pole particle. In other words, when the last particle in the string is displaced upwards, there will be no adjoining pole particle to pull it down. As a result, the upward displacement of the incident pulse is not reversed the in the reflected pulse. Similarly, if the incident pulse has a downward displacement the reflected pulse will also demonstrate a downward displacement. . Inversion is not observed in free end reflection.

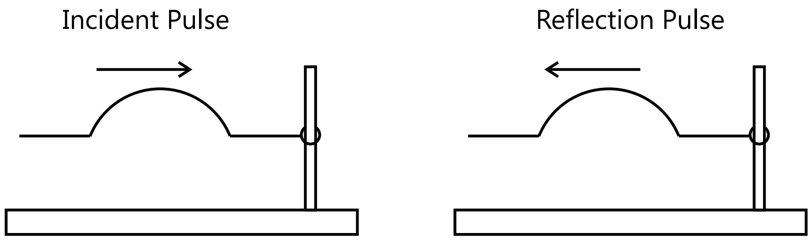


Figure 11.18

10. REFLECTION AND TRANSMISSION OF A PULSE ACROSS A BOUNDARY

10.1 Reflection and Transmission of a Pulse across a Boundary from Less to More Dense

A pulse exhibits two behaviors upon reaching the boundary.

(a) A part of the energy transmitted by the incident pulse is reflected and returns towards the hand-held end of a thin string. The pulse that returns to the hand-held end after bouncing off the boundary is known as the reflected pulse.

(b) A part of the energy transmitted by the incident pulse is transmitted into the thick string. The disturbance that continues moving to the right is known as the transmitted pulse.

In a wave traveling from a less dense to a denser medium a part of the incident pulse will be reflected off the boundary of the less dense string while another part will be transmitted across the boundary of the thin string into the new medium (thick string). The pulse that moves into the new medium is the transmitted pulse and is not inverted. The pulse that is reflected off the boundary of the thinner string is called the reflected pulse is inverted.

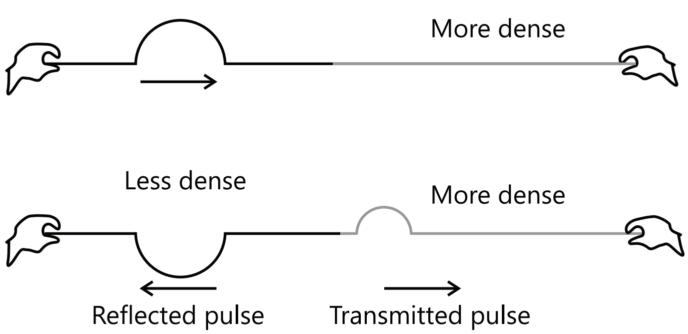


Figure 11.19

10.2 Reflection and Transmission of a Pulse across a Boundary from More to Less Dense

Here, the transmitted pulse moves through the less dense string/medium, while the reflected pulse travels through the denser string/medium. The transmitted pulse travels faster and has larger wavelength than the reflected pulse. However, the speed and wavelength of the reflected pulse are same as the that of the incident pulse.

Here, a part of the incident pulse will be reflected off the boundary of the denser string/medium and part will transmitted across the boundary of the denser string/medium into the less dense string/medium. There is no inversion, whatsoever.

A wave travelling from a more dense to a less dense medium

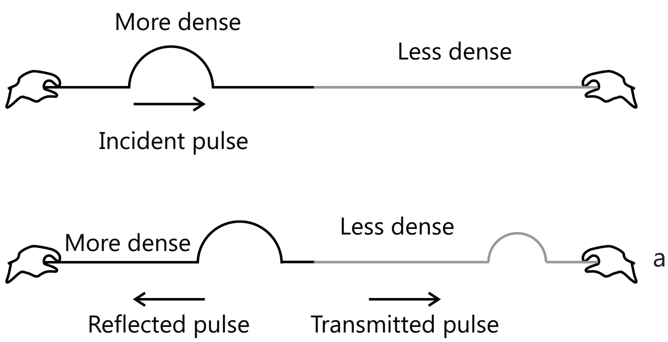


Figure 11.20

MASTERJEE CONCEPTS

- The wave speed and the wavelength are always greatest in the least dense string/medium.
- The wave frequency remains constant even when crosses the boundary.
- When moving from less dense string/medium to denser string/medium, the reflected pulse gets inverted.
- The amplitude of the incident pulse is always greater than that of the reflected pulse.

Anand K (JEE 2011, AIR 47)

11. STANDING WAVES

Standing wave, also called a stationary wave, is combination of two waves moving in opposite directions, each having the same amplitude and frequency. . The manner of this interference makes it appear as if some points along the medium are standing still. For this reason, this wave pattern is referred to as the standing wave pattern. Let us assume that two waves of equal amplitude and frequency propagate towards each other along a string. The equation of two waves are given by $y_1 = A\sin(\omega t - kx)$ and $y_2 = A\sin(\omega t + kx + \delta)$.

To understand these waves, let us discuss the special case when $\delta = 0$.

The displacements of the particles of the string consequent to the interference are given by the principle of superposition as $y = y_1 + y_2 = A\,[\sin(\omega t\text{-}kx) + \sin(\omega t+kx) = 2A \sin\omega t \cos kx$ or $y = (2A \cos kx) \sin \omega t$ ... (xix)

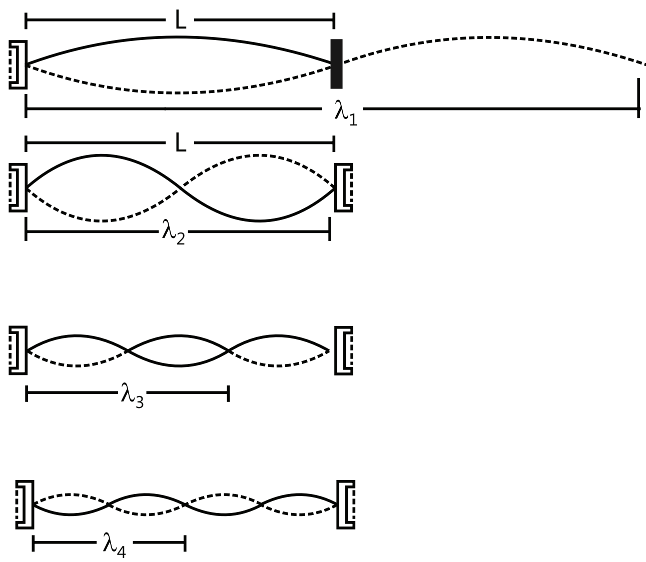


Figure 11.21

11.1 Nodes and Antinodes

As discussed earlier, the manner of interference of standing wave patterns is such that there are points along the medium that appear to be stationary. These points are referred to as nodes or points of no displacement. There are other points along the medium that undergo v the maximum displacement during each vibrational cycle of the standing wave. These points along the medium are called antinodes, as they represent the other extreme in the standing wave pattern. A standing wave pattern always has nodes and antinodes appearing alternatively in them

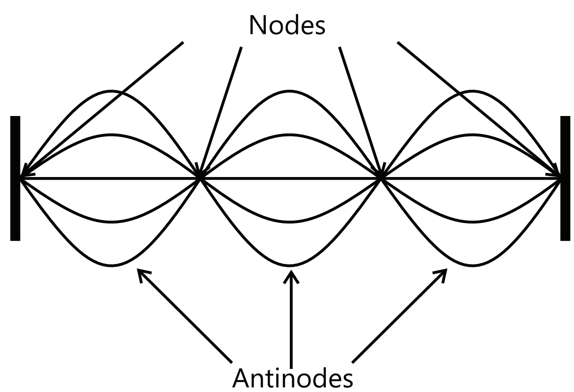


Figure 11.22

MASTERJEE CONCEPTS

Nodes and antinodes are quite different from crests and troughs. In a traveling wave, there points of large upward and downward displacements, referred to as the crest and trough of the wave. However, an antinode refers to a point of the string that remains stationary or appear to be stationary.

GV Abhinav (JEE 2012, AIR 329)

11.2 Differences Between Standing Waves and Traveling Waves

Standing Wave	**Traveling Wave**
The disturbance produced in a region appears stationary.	The disturbance produced in a region is transmitted with a definite velocity
Different particles move with different amplitudes	The motion of all particles are similar in nature
The particles at node always remain at rest	There is no particle which always remains at rest
All particles cross their mean positions together	At no point all the particles are at mean positions together
All the particles between two successive nodes reach their extreme positions together, thus moving in phase.	The phases of nearby particles are always different
The energy of one region is always confined to that region	Energy is transmitted from one region of space to other

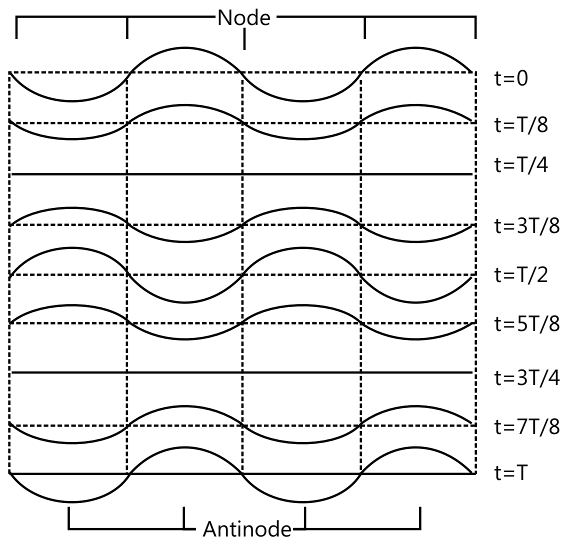


Figure 11.23

Illustration 12: The interference of two waves with equal amplitudes and frequencies traveling in opposite directions produces a standing wave having the equation Y = A cos kx sin ωt in which

A = 1.0 mm, k = 1.57 cm^{-1} and ω = 78.5 s^{-1}

(a) Find the velocity of the component traveling waves.

(b) Find the node closest to the origin in the region x > 0.

(c) Find the antinode closest to the origin in the region x > 0.

(d) Find the amplitude of the particle at x = 2.33 cm. **(JEE ADVANCED)**

Sol: Here the two waves of same amplitude and frequency interfere with each other to form the standing waves, the velocity of the resultant wave will be $v = \frac{\omega}{k}$ where ω is the angular frequency of the wave and k is the wave number. The distance of the node from the origin is given by $kx = \frac{n\pi}{2}$. And distance of antinode from origin is given by $kx = n\pi$.

(a) The standing wave is formed by the superposition of the waves

$y_1 \frac{A}{2}\sin(\omega t - kx)$ and $y_2 \frac{A}{2}\sin(\omega t + kx)$. The wave velocity (magnitude) of the waves is $v = \frac{\omega}{k} = \frac{78.5\ s^{-1}}{1.57\ cm^{-1}} = 50\ cms^{-1}$

(b) For a node, cos kx = 0

The smallest position x satisfying this relation is given by $kx = \frac{\pi}{2}$ or, $x = \frac{\pi}{2k} = 1cm$

(c) For an antinode, |cos kx| = 1 or cos kx = ± 1

$Kx = 0,\ \pi, 2\pi, \ldots\ldots n\pi \Rightarrow x \neq 0, x_{min} = \frac{\pi}{k} = \frac{3.14}{1.57} = 2cm$

(d) The amplitude of vibration of the particle at x is given by |A cos kx|

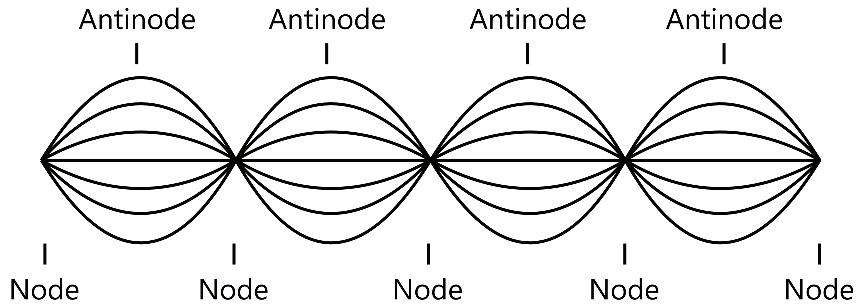


Figure 11.24

For the given point, $kx = (1.57cm^{-1})(2.33cm) = \frac{7}{6}\pi = \pi + \frac{\pi}{6}$

Thus, the amplitude will be $(1.0\ mm)\ |\ \cos(\pi + \pi/6)\ | = \frac{\sqrt{3}}{2} mm = 0.86 mm$

11.3 Standing Waves on a String Fixed at Both Ends (Qualitative Discussion)

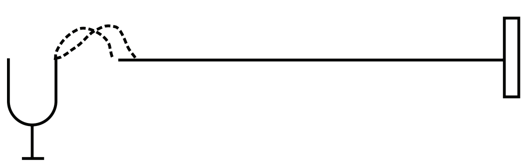


Figure 11.25

Let us take the example of string fixed at both the ends –- one end to a wall and the other end tied to a tuning fork. The tuning fork vibrates longitudinally with a small amplitude producing sine waves of amplitude A which travel along the string towards the wall. The said wave then gets reflected and travels toward the fork. This wave, being reflected from a fixed end, will be an inverted wave.. These waves are again hit the fork back and as the fork is heavy and vibrates longitudinally with a small amplitude, it acts like a fixed end and the waves reflected from the fork get inverted again. Therefore, the wave produced directly by the fork initially and the twice-reflected wave have same shape, though the twice-reflected wave has already travelled a length 2L.

Let us assume that the length of the string is 2L=λ. The wave moving from the tuning fork to the wall and the wave reflected back from the wall to the tuning fork interfere constructively and the resultant wave that proceeds towards the wall has an amplitude 2A. This wave of amplitude 2A is again reflected back by the wall and then again reflected by the fork. Now, this twice-reflected wave again interfaces constructively with the new incident wave and a wave of amplitude 3A is produced. Thus, the amplitude keeps progressing. The string gets energy from the vibrating and the amplitude builds up. Same arguments hold if 2L is any integral multiple of λ that is $L=n\lambda/2$, where n is an integer.

However, in the above discussion, we have not factored in any loss of energy due to air viscosity or due to the inflexibility of the string. In the steady state, waves of invariable amplitude will be present on the string from left to right as well as from right to left. These opposing waves will produce standing waves on the string. Nodes and antinodes will be formed along the string and there will be large amplitudes of vibration at the antinodes. We can then say that the string is in resonance with the fork. The condition, $L=n\lambda/2$, for such a resonance may be stated in a different way. We have from equation (9), $\upsilon = \nu\lambda$ or $\lambda = \upsilon/\nu$

The condition for resonance is, therefore,

$$L = n\frac{\lambda}{2} \quad \text{or} \quad L=\frac{n\upsilon}{2\nu} \quad \text{or} \quad \nu = \frac{n\upsilon}{2L} = \frac{n}{2L}\sqrt{F/\mu} \quad \text{... (i)}$$

The lowest frequency with which a standing wave can be set up in a string fixed at both the ends is thus

$$\nu_o = \frac{1}{2L}\sqrt{F/\mu} \quad \text{... (ii)}$$

This is called the fundamental frequency of the string. All other possible frequencies of standing waves are integral multiples of this fundamental frequency. Equation (xx) gives the natural frequencies, normal frequencies, or resonant frequencies.

Illustration 13: Shown in the Fig 11.26 is a wire with a length of 50 cm and a mass of 20 g. It supports a mass of 1.6 kg. Find the fundamental frequency of the portion of the string between the wall and the pulley.

Take $g=10ms^{-2}$. **(JEE ADVANCED)**

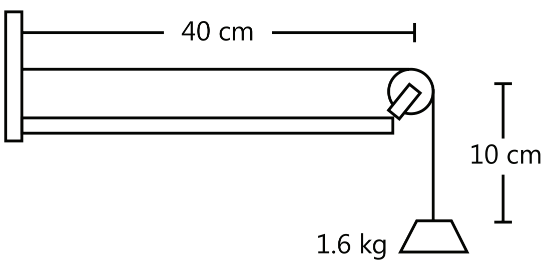


Figure 11.26

Sol: The string is subjected to uniform tension due to weight of the block of mass 1.6 kg. The fundamental frequency of the string between the fixed support and pulley is given by $\nu_o = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$ where μ is the mass per unit length of string.

The tension in the string is F = (1.6kg) ($10ms^{-2}$) = 16N.

The linear mass density is $\mu = \frac{20g}{50cm} = 0.04 kgm^{-1}$

The fundamental frequency is $\nu_o = \frac{1}{2L}\sqrt{\frac{F}{\mu}} = \frac{1}{2\times(0.4m)}\sqrt{\frac{16N}{0.04\,kgm^{-1}}}$ = 25 Hz

11.4 Analytical Treatment of Vibration of a String Fixed at Both Ends

Let us assume a string of length L which is kept fixed at the ends x= 0 and x= L. For certain wave frequencies, standing waves are set up in the string. Due to the repeated reflection of the wave at the ends and the damping effects, waves going in the positive direction x interfere to give a resultant wave $y_1 = A\sin(kx-\omega t)$. Similarly, the waves going in the negative direction x interfere to give the resultant wave $y_2 = A\sin(kx+\omega t+\delta)$. As a result, the displacement of the particle of the string at position x and at time t is given by the principle of superposition as $y = y_1 + y_2 = A\sin(kx-\omega t) + \sin(kx+\omega t+\delta)$

$$= 2A\sin(kx - \frac{\delta}{2})\cos(\omega t + \frac{\delta}{2}) \qquad \text{... (i)}$$

If standing waves are formed, the ends x = 0 and x = L must be nodes because they are kept fixed. Thus, we have the boundary conditions y= 0 at x=0 for all t and y=0 at x=l for all t.

The first boundary condition is satisfied by equation (i) if $\sin\frac{\delta}{2} = 0$, or $\delta = 0$.

Equation (i) then becomes $y = 2A\sin kx \cos\omega t$ ... (ii)

The second boundary condition will be satisfied if

$\sin kL = 0$ or $kL = n\pi$, where n=1, 2, 3, 4, 5,.......

or $\frac{2\pi L}{\lambda} = n\pi$ or $L = \frac{n\lambda}{2}$ ... (iii)

If the length of the string is an integral multiple of $\lambda/2$, standing waves are produced.

Again writing $\lambda = \upsilon T = \frac{\upsilon}{v}$, equation (xxv) becomes $v = \frac{n\upsilon}{2L} = \frac{n}{2L}\sqrt{F/\mu}$

Which is same as equation (xx). The lowest possible frequency is $v_o = \frac{\upsilon}{2L} = \frac{1}{2L}\sqrt{F/\mu}$ ... (iv)

This is the fundamental frequency of the string. The other natural frequencies with which standing wave can be formed on the string are

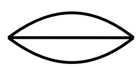
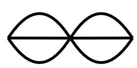
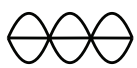
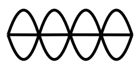
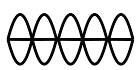
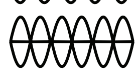
Harmonic	Pattern	No. of Loops	Length-Wavelength relationship
1st		1	L = 1/2 · λ
2nd		2	L = 2/2 · λ
3rd		3	L = 3/2 · λ
4rd		4	L = 4/2 · λ
5th		5	L = 5/2 · λ
6th		6	L = 6/2 · λ

Figure 11.27

$v_1 = 2v_0 = \frac{2}{2L}\sqrt{F/\mu}$ 1st overtone, or 2rd harmonic,

$v_2 = 3v_0 = \frac{3}{2L}\sqrt{F/\mu}$ 2nd overtone, or 3rd harmonic,

$v_3 = 4v_0 = \frac{4}{2L}\sqrt{F/\mu}$ 3rd overtone, or 4rd harmonic, etc.

In general, any integral multiple of the fundamental frequency is a valid frequency. These higher frequencies are called overtones. Thus, $v_1 = 2v_0$ is the first overtone, $v_2 = 3v_0$ is the second overtone, etc. An integral multiple of a frequency is called its harmonic. Thus, for a string fixed at both the ends, all the overtones are harmonics of the fundamental frequency and vice-versa.

11.5 Vibration of a String Fixed at One End

If a string is set up in such a way that one end of it remains fixed while the other end is free to move in a transverse direction, standing waves can be produced. The free end can be created by connecting the string to a very light thread. If the vibrations of the "correct" frequency are produced by the source, standing waves are produced. Assuming end x=0 is fixed and x = L is free, the equation is again given by $y = 2A\sin kx\cos\omega t$ which is the same as equation (xxii), with the boundary condition that x = L is an antinode. The boundary condition that x = 0 is a node is automatically satisfied by the above equation as it is fixed. For x = L to be an antinode, $\text{Sin } kL = \pm 1$

or $kL = \left(n+\frac{1}{2}\right)\pi$ or $\frac{2\pi L}{\lambda} = \left(n+\frac{1}{2}\right)\pi$ or $\frac{2Lv}{\upsilon} = n+\frac{1}{2}$ or $v = \left(n+\frac{1}{2}\right)\frac{\upsilon}{2L} = \frac{n+\frac{1}{2}}{2L}\sqrt{F/\mu}$ ... (i)

These are the normal frequencies of vibration. The fundamental frequency is obtained when n = 0,

i. e., $\nu_o = \upsilon/4L$

The overtone frequency are $\nu_1 = \frac{3\upsilon}{4L} = 3\nu_o$

$\nu_2 = \frac{5\upsilon}{4L} = 5\nu_o$, $\nu_3 = \frac{7\upsilon}{4L} = 7\nu_o$, etc

It can be seen that all the harmonics of the fundamental frequency are not the valid frequencies for the standing waves. Only the odd harmonics are the overtones. The string shapes for some of the normal modes are shown in Fig 11.28.

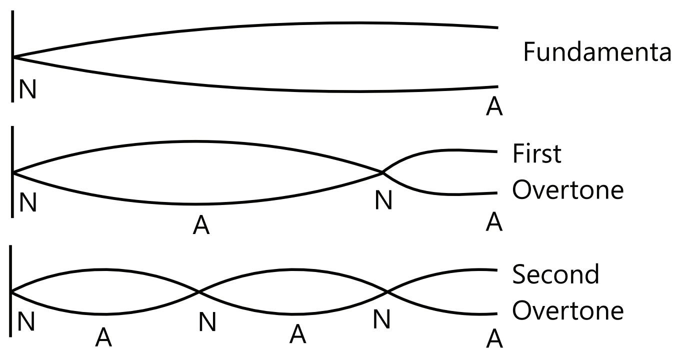


Figure 11.28

Illustration 14: A string is vibrating up and down as the fifth harmonic and completes 21 vibrational cycles in 5 seconds. The length of the string is 8.2 meters. Determine the frequency, period, wavelength and speed for this wave. **(JEE MAIN)**

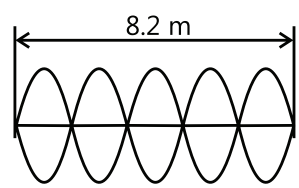


Figure 11.29

Sol: The frequency of the wave is $f = \frac{\text{number of cycles produced}}{\text{total time}}$. The time period of wave $T = \frac{1}{f}$. When string is vibrating in fifth harmonics, then 2L = 5λ. The wave velocity is v = f λ.

Given: L = 8.2 m and 21 cycles in 5 seconds. The frequency here refers to the number of back-and-forth movements of a point on the string and is measured as the number of cycles per unit of time. In this case, it is f = (21 cycles)/(5 seconds) = 4.2 Hz

The period is the reciprocal of the frequency. T = 1/ (4.2 Hz) = 0.238 s.

The wavelength of the wave is correlated to the length of the rope. For the fifth harmonic as shown in the picture, the length of the rope is equivalent to five halves of a wavelength. That is, $L = \frac{5}{2}\lambda$ where λ is the wavelength. Rearranging and substituting the equation gives the following results:

$\lambda = (2/5) \times L = (2/5) \times (8.2\text{ m}) = 3.28\text{ m}$

The wavelength and frequency wave can be used to calculate the speed of a wave using the wave equation

$V = f\lambda = (4.2\text{Hz}).(3.28\text{m}) = 13.8\text{m/s}$

12. LAWS OF TRANSVERSE VIBRATIONS OF A STRING

For a string fixed at both ends, the fundamental frequency of vibration is given by equation (ix). The statements known as "Laws of transverse vibrations of a strings" can be derived from equation (ix).

12.1 Law of Length

Tension and mass per unit length of the string remaining the same, the fundamental frequency of vibration of a string (fixed at both ends) is inversely proportional to the length of the string.

$\nu \propto 1/L$ if F and μ are constants.

12.2 Law of Tension

The length and the mass per unit length of the string remaining the same, the fundamental frequency of a string is proportional to the square root of its tension. $\nu \propto \sqrt{F}$ L if F and μ are constants.

12.3 Law of mass

The length and the tension remain the same, the fundamental frequency of a string is inversely proportional to the square root of the linear mass density, i.e., mass per unit length.

$$\nu \propto \frac{1}{\sqrt{\mu}} \text{ if L and F are constants.}$$

These above laws can be experimentally studied with an apparatus called sonometer.

12.4 Sonometer

A sonometer is an apparatus that is used to study the transverse vibrations of strings. It is also called the monochord because it often has only one string. It consists of a rectangular wooden box with two fixed bridges near the ends, with a pulley fixed at one end. A string is fixed at one end, which is then run over the bridges and the pulley, and then attached to a weight holder hanging below the pulley. Additional weights can be added to the holder to increase the tension in the wire. A third, movable bridge, can be placed under the string to change the length of the vibrating section of the string. This device demonstrates the relationship between the frequency of the sound produced when a string is plucked and the tension, length, and mass per unit length of the string. These relationships are referred to as Mersenne's law after Marin Mersenne (1588–1648), who studied and formulated them. For small amplitude vibration, the frequency is proportional to:

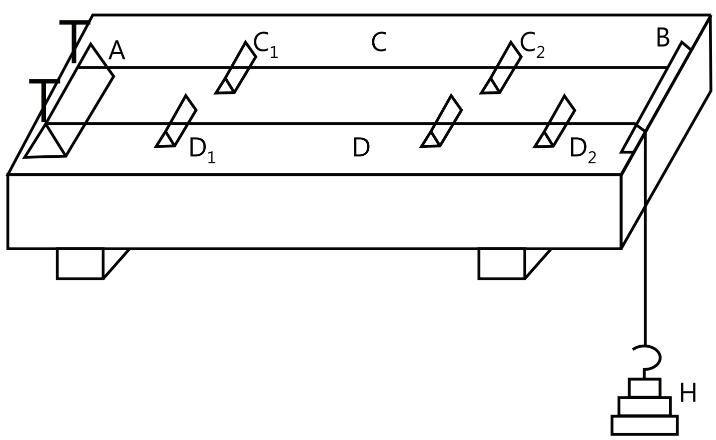


Figure 11.30

(a) The square root of the tension of the string

(b) The reciprocal of the square root of the linear density of the string,

(c) The reciprocal of the length of wire of sonometer

Illustration 15: Resonance is obtained in a sonometer experiment when the experimental wire with a length of 21 cm between the bridges is excited by a tuning fork of frequency 256 Hz. If a tuning fork of frequency 384 Hz is used, what should be the length of the experimental wire to get the resonance? **(JEE MAIN)**

Sol: For sonometer wire the ratio of lengths of vibrating string is $\frac{\ell_1}{\ell_2} = \frac{\nu_2}{\nu_1}$.

By the law of length, $\ell_1\nu_1 = \ell_2\nu_2$ or $\ell_2 = \frac{\nu_1}{\nu_2}\ell_1 = \frac{256}{384}\times 21\text{cm} = 14\,\text{cm}$

13. TRANSVERSE AND LONGITUDINAL WAVES

When there is a disturbance at the source in a string, it causes displacement of the particles of the string. The direction of such displacements is perpendicular to the direction of the propagation of the wave. Such waves are called transverse waves. The wave on a string is a transverse wave.. Light waves are also an example of transverse waves. Here, the value of the electric field changes with space and time and the changes are propagated in space. The direction of the electric field is perpendicular to the direction of propagation of light when light travels in free space.

Sound waves are an example of non-transverse waves. The particles of the medium are carried along the direction of propagation of sound. We shall study in some detail the mechanism of sound waves in the next chapter. If the displacement produced by the passing wave is along the direction of the wave propagation, the wave is called a longitudinal wave. Sound waves are longitudinal.

13.1 Compression and Rarefaction

A longitudinal wave consists of compressions and rarefactions. Those regions in a longitudinal wave where particles are clustered together are compressions. Conversely, those regions where the particles are furthest apart are called rarefactions.

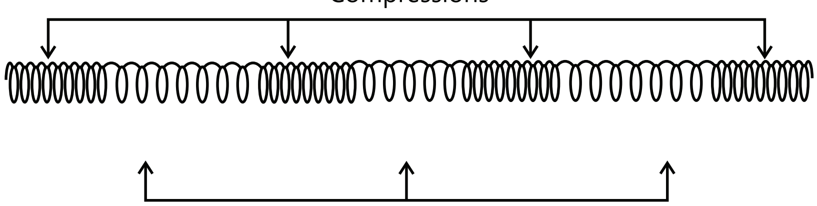


Figure 11.31

Illustration 16: A sonometer wire has a length of 100 cm and a fundamental frequency of 330 Hz. Find

(a) The velocity of propagation of transverse waves along the wire and

(b) The wavelength of the resulting sound in air if velocity of sound in air is 330 m/s. **(JEE ADVANCED)**

Sol: When sonometer wire is set to vibrate in its fundamental frequency, then wavelength is $\lambda=2L$, the wave velocity is $v = f\lambda$ where f is the frequency of oscillation.

(a) In case of transverse vibration of string for fundamental mode:

$L = (\lambda / 2)$, i.e., $\lambda = 2L = 2\times l = 2m$

i.e., the wavelength of transverse wave propagation on string is 2 m. Since the frequency of the wire is given to be 330 Hz, so from $\upsilon = f\lambda$, the velocity of transverse waves along the wire will be

$V_{wire} = 330\times 2 = 660\,m/s$

i.e., for transverse mechanical waves propagation along the wire, Hz,m and m/s

(b) Here vibration wire will act as source and produce sound, i.e., longitudinal waves in air. Now as frequency does not change with change in medium so Hz and as velocity in air is given to be = 330 m/s so from $\upsilon = f\lambda$;

$\lambda_{air} = (V_{air} / f) = (330/330) = 1m$

i. e., for sound (longitudinal mechanical waves) in air produced by vibration of wire (body),

f = 330Hz, $\lambda = 2m$ and $v = 330m/s$

14. POLARIZATION OF WAVES

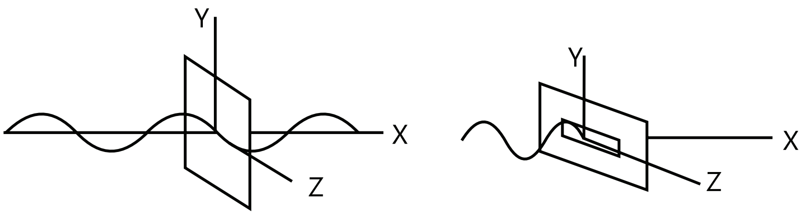


Figure 11.32

Let us assume that we have a cardboard with a slit in it through which a stretched string is passed such that the card is placed in a perpendicular position to the string. (See Fig 11.32). If we take the string as the X axis, the cardboard will be in Y-Z plane. Now we generate a wave along the X axis such that the particles of the string are displaced in direction Y as the wave passes. If the slit in the cardboard is also aligned along the Y axis, the portion of the string in the slit can vibrate freely in the slit and the wave will pass through the slit. Now, if we turn the cardboard by 90^0 in its plane, the slit will be aligned along the Z axis. As the wave reaches the slit, the portion of the string in the slit tries to move along the Y axis but the narrow slit on the cardboard becomes an obstruction. Consequentially, the wave is not able to pass through the slit. However, if the slit is inclined to the Y axis at a different other angle, only a part of the wave is transmitted and in the transmitted wave the disturbance is produced parallel to the slit. The same experiment can be conducted with two chairs as shown in the Fig 11.33. If the displacement produced is always along a fixed direction, then the wave is said to be linearly polarized in that direction. The examples considered in this chapter are linearly polarized in y direction. By the same token, a wave that produces a displacement along the z direction, is a linearly polarized wave, polarized in z-direction. Its equation is given by $z = A\sin\omega(t - x/\upsilon)$ .Linearly polarized waves are referred to as plane polarized. In the event that each particle of a string moves in a small circle when the wave is propagated, the wave is called circularly polarized. If each particle goes in ellipse, the wave is called polarized. If the particles are move randomly in the plane perpendicular to the direction of propagation, the wave is called un-polarized.

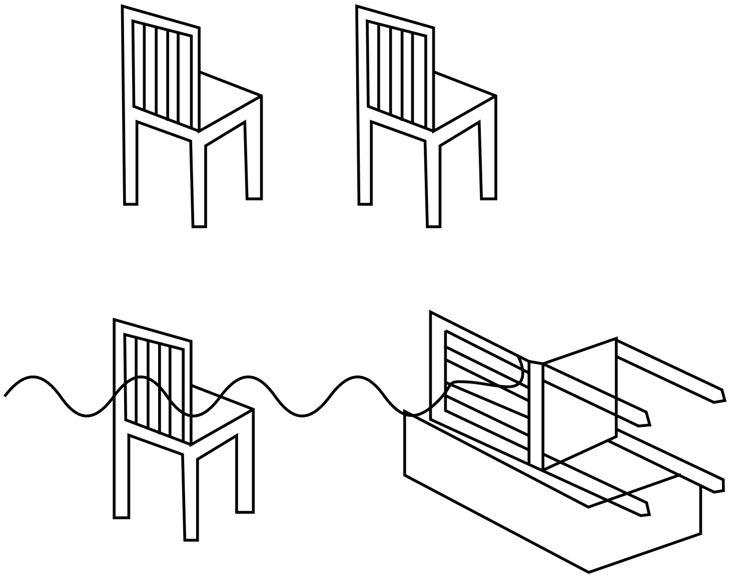


Figure 11.33

PROBLEM-SOLVING TACTICS

1. Understanding and remembering all formulae is the key to solving problems in these sections. If the relation between the given quantities and the questions asked is known, it will be easy to solve most of the problems. All the quantities discussed in this topic are in some sense related to each other.
2. The concept of reflection (of waves) can be encapsulated in a single point: "Inversion- Reflected wave will invert only when it encounters a denser medium. And transmitted wave will never invert." If this much is clear, one can easily identify the case in every question.
3. Waves must always be understood in the context of transfer of energy rather than as just some function of x and t for better understanding of physics.
4. For questions pertaining to the derivation of the wave equation, one can begin easily with only the x part and subsequently add or subtract vt from x depending on the direction of velocity.
5. Most questions related to velocity and energy appear complicated due to the introduction of the usual Newton mechanics. This should, however, be treated just as some additional information to calculate tension in the string (e.g., Pulley systems).

FORMULAE SHEET

S. No	Term	Description
1	Wave	It is a disturbance or variation traveling through a medium due to the repeated undulating motion of particles of the medium through their equilibrium position. Examples are sound waves travelling through an intervening medium, water waves etc.
2	Mechanical waves	Waves that are propagated through a material medium are called MECHANICAL WAVES. These are governed by Newton's Law of Motion. Sound waves are mechanical waves propagated through the atmosphere from a source to the listener and it requires a medium for its propagation.
3	Non mechanical waves	Waves which are not propagated through a material medium. Eg: light waves, EM waves.
4	Transverse wave	These are waves in which the displacements or oscillations are perpendicular to the direction of propagation of the wave.
5	Longitudinal wave	Longitudinal wave waves in which the displacement or oscillations in medium are parallel to the direction of propagation of wave. Example: sound waves
6	Equation of harmonic wave	At any time t, displacement y of the particle from its equilibrium position as a function of the coordinate x of the particle is $y(x, t) = A\sin(\omega t - kx)$ where, A is the amplitude of the wave, K- is the wave number ω is angular frequency of the wave and $(\omega t\text{-}kx)$ is the phase
7	Wave number	Wavelength λ and wave number k are related by the relation $k = 2\pi/\lambda$

8	Frequency	Time period T and frequency f of the wave are related to ω by $\omega/2\pi = f = 1/T$
9	Speed of wave	Speed of the wave is given by $v = \omega/k = \lambda/T = \lambda f$
10	Speed of a transverse wave	The tension and the linear mass density of a stretched string, and not the frequency, determines the speed of a transverse wave i.e., $v=\sqrt{\frac{T}{\mu}}$ T = Tension in the string μ = Linear mass density of the string.
11	Speed of longitudinal waves	Speed of longitudinal waves in a medium is given by $v = \sqrt{\frac{B}{\rho}}$; B = bulk modulus; ρ = Density of the medium speed of longitudinal waves in ideal gas is $v = \sqrt{\frac{\gamma P}{\rho}}$ P = Pressure of the gas, ρ = Density of the gas and $\gamma = C_p / C_V$
12	Principle of superposition	It states that when two or more waves of same type come together at a single point in space, the total displacement at that point is equal to the sum of the displacements of the individual waves. It is given by $y=\sum y_i(x,t)$
13	Interference of waves	Two sinusoidal waves traveling in the same direction interfere to produce a resultant sinusoidal wave traveling in that direction if they have the same amplitude and frequency, with resultant wave given by the relation $y'(x,t)=[2A_m \cos(u/2)]\sin(\omega t - kx + u/2)$ where u is the phase difference between two waves. If u = 0, then interference would be fully constructive. If u = π, then waves would be out of phase and the interference would be destructive.
14	Reflection of waves	An incident wave encountering a boundary gets reflected. If an incident wave is represented by $y_i(x,t) = A\sin(\omega t - kx)$ then reflected wave at rigid boundary is $y_r(x,t) = A\sin(\omega t + kx + \pi) = -A\sin(\omega t + kx)$ And for reflections at open boundary, the reflected wave is given by $y_r(x,t) = A\sin(\omega t + kx)$
15	Standing waves	When two identical waves moving in opposite directions meet, the interference produces standing waves. The particle displacement in standing wave is given by $y(x,t) = [2A\sin(kx)]\sin(\omega t)$. The amplitude of standing waves is different at different point i.e., at nodes amplitude is zero and at antinodes amplitude is maximum or equal to sum of amplitudes of constituting waves.

16	Normal modes of stretched string	Frequency of transverse waves in a stretched string of length L and fixed at both the ends is given by f = nv /2L where n = 1, 2, 3........ The above relation gives a set of frequencies called normal modes of oscillation of the system. Mode n=1 is called the fundamental mode with frequency f_1= v/2L. Second harmonic is the oscillation mode with n = 2 and so on. Thus the string has infinite number of possible frequency of vibration which are harmonics of fundamental frequency f_1 such that $f_n = nf_1$.

Solved Examples

JEE Main/Boards

Example 1: The length of a wave propagated on a long stretched string is taken as the positive x axis. The wave equation is given by

$y = y_0 e^{-\left(\frac{t}{T}-\frac{x}{\lambda}\right)^2}$ where $y_0 = 4\,\text{mm}$, $T = 1.0\,\text{s}$ and $\lambda = 4\,\text{cm}$.

(a) Find the velocity of the wave.

(b) Find the function finding the displacement of the particle at x = 0.

(c) Find the function giving the shape of the string at = 0.

(d) Plot the shape of the string at t =0.

(e) Plot the shape of the string at t=5s.

Sol: The wave moves having natural frequency of ν and wavelength λ has velocity $V = \nu\lambda$. As the frequency is $\nu = \frac{1}{T}$ the velocity of the wave is then $V = \frac{\lambda}{T}$.

(a) The wave equation may be written as

$y = y_0 e^{-\frac{1}{T}\left(t-\frac{x}{\lambda/t}\right)^2}$

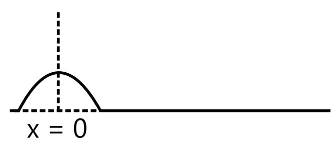
Comparing with the general equation we see that

$$\nu = \frac{\lambda}{1.0\,\text{s}} = 4\,\text{cm} = 4\,\text{cm/sec}$$

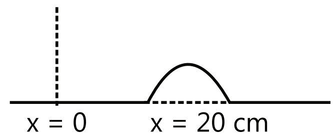
(b) Putting x = 0 in the given equation $f(t) = y_0\, e^{-(t/T)^2}$ … (i)

(c) Putting t = 0 in the given equation $g(t) = y_0\, e^{-(x/\lambda)^2}$ ... (ii)

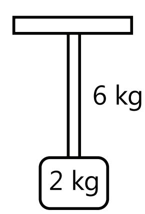
(d)



(e)



Example 2: The dimensions of a uniform rope are as follows: length 12 m, mass 6 kg. The rope hangs vertically from a rigid support with a slab of a mass of 2 kg is attached to the free end of the rope. If a transverse pulse of wavelength 0.06 m is transmitted from the free end of the rope, what is the wavelength of the pulse when it reaches the top of the rope?



Sol: The wave velocity will be $V = \nu\lambda = \sqrt{\frac{F}{\mu}}$ where F is the tension in string at a point and μ is mass per unit length of the string. As F is varying along the length of the rope so the velocity will vary along the length of the rope. As source frequency is constant λ will vary.

We have, $V = \nu\lambda$

Or, $\sqrt{\frac{F}{\mu}} = \nu\lambda$ or $\frac{\sqrt{F}}{\lambda} = \nu\sqrt{\mu}$

Since the frequency of the wave pulse is dependent only on the frequency of the source, it will be consistent