10. GRAVITATION

1. INTRODUCTION

Have you ever wondered whether we would still be studying about with Gravitation if a stone had fallen on Newton's head instead of an apple? Anyways, the real question is, why does an apple fall down rather than go upward?

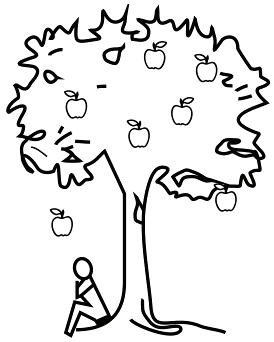


Figure 10.1

2. NEWTON'S LAW OF UNIVERSAL GRAVITATION

"Every particle of matter in the universe attracts every other particle with a force equal to the product of masses of particles and inversely proportional to the square of the distance between them"

If m_1 and m_2 are two point masses separated by a distance r, the gravitational force of attraction F is given by

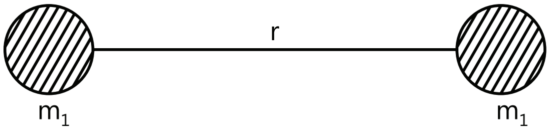


Figure 10.2

$$F \propto \frac{m_1 m_2}{r^2}$$

$$F = \frac{G m_1 m_2}{r^2}$$

Where G is a constant and is called the Universal gravitational constant.

Magnitude (and unit) of G : 6.67×10^{-11} Newton. m^2 / kg^2

Dimension of G : $M^{-1}L^{3}T^{-2}$

MASTERJEE CONCEPTS

The direction of force F is independent of the medium, not affected by the presence of the other bodies and acts along the line joining the two particles.

If two persons come very close to each other such that the distance between them is almost 0, the two persons should experience a high force of attraction. Observe keenly the value of G. It's of order -11.

The Universal gravitational constant G is an experimental value calculated by Cavendish 71 years after the law was formulated.

Always remember Gravitational Force is conservative in nature i.e. work done doesn't depend on the path taken and depends only on the end points.

Vaibhav Gupta (JEE 2009, AIR 54)

Illustration 1: Two particles of masses 1.0 kg and 2.0 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial accelerations of the two particles.

(JEE MAIN)

Sol: The force of mutual gravitation acting on particles is $F=\frac{Gm_1m_2}{r^2}$. As the particle are accelerating under the force of gravitation, the acceleration is obtained using Newton's laws of motion.

The force of gravitation exerted by one particle on the other is

$$F=\frac{Gm_1m_2}{r^2}=\frac{6.67\times10^{-11}\frac{N-m^2}{kg^2}\times(1.0kg)\times(2.0kg)}{(0.5m)^2}=5.3\times10^{-10}N.$$

The acceleration of 1.0 kg particle is $a_1=\frac{F}{m_1}=\frac{5.3\times10^{-10}N}{1.0kg}=5.3\times10^{-10}ms^{-2}$

This acceleration is towards the 2.0 kg particles. The acceleration of the 2.0 kg particle is

$$a_2=\frac{F}{m_2}=\frac{5.3\times10^{-10}N}{2.0kg}=2.65\times10^{-10}ms^{-2}$$

This acceleration is towards the 1.0 kg particle.

Illustration 2: Spheres of the same material and same radius r are touching each other. Show that gravitational force between them is directly proportional to r^4. **(JEE MAIN)**

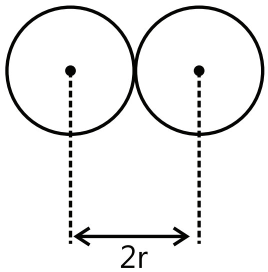


Figure 10.3

Sol: The force of gravitation is directly proportional to the masses of the spheres. As the spheres are having the same masses, and mass $m\propto V\Rightarrow m\propto r^3$ thus the proportionality between the force and distance is easily established.

As the spheres are made of same material, and density so the mass of each sphere is $m_1=m_2=$ (volume) (destiny)

$$=\left(\frac{4}{3}\pi r^3\right)\rho$$

$$F=\frac{Gm_1m_2}{(2r)^2}=\frac{G\left(\frac{4}{3}\pi r^3\right)\left(\frac{4}{3}\pi r^3\right)\rho^2}{4r^2}\qquad\text{or}\qquad F\propto r^4$$

Illustration 3: Three particles each of mass m, are located at the vertices of an equilateral triangle of side a. At what speed will they move if they all revolve under the influence of their gravitational force of attraction in a circular orbit circumscribing the triangle while still preserving the equilateral triangle? **(JEE MAIN)**

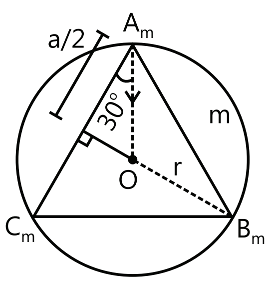


Figure 10.4

Sol: The net force of gravitation on any one particle is due to other two particles. This gravitational force provides the necessary centripetal force to the particles to move in the circular orbit around the equilateral triangle.

$$\vec{F}_A=\vec{F}_{AB}+\vec{F}_{AC}=2\left[\frac{GM^2}{a^2}\right]\cos30^\circ=\left[\frac{GM^2}{a^2}\sqrt{3}\right]$$

$$r=\frac{a}{\sqrt{3}},\quad \text{Now } \frac{mv^2}{r}=F;\ \text{Or}\quad \frac{mv^2\sqrt{3}}{a}=\frac{GM^2}{a^2}\sqrt{3};\quad \therefore v=\sqrt{\frac{GM}{a}}$$

3. GRAVITATIONAL FIELD

How would a particle interact with the surrounding or with other particles?

Every particle creates a field and when the other particle comes in to this particle's field, there would be an interaction between the particles.

The intensity of the field i.e. how intensely would it attract another particle in its field is called Gravitational field intensity or Gravitational field strength $\vec{E}$. It is defined as the force experienced by a unit mass placed at a distance r due to mass M, i.e. $\vec{E}=\frac{\vec{F}}{M}$

MASTERJEE CONCEPTS

Always remember, it is a vector quantity and should be added vectorially when calculating Gravitational field intensity at a point by one or more masses.

Vaibhav Krishnan (JEE 2009, AIR 22)

4. GRAVITATIONAL FIELD INTENSITY

(a) Due to a point mass M:

$$F=\frac{GMm}{r^2};\qquad E=\frac{F}{m}=\frac{GM}{r^2};\qquad E=\frac{GM}{r^2}$$

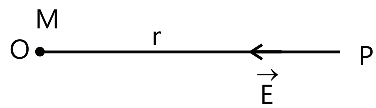


Figure 10.5

(b) Due to uniform ring of Mass M and radius a on its axis.

Consider any particle of mass dm on the ring, say at point A. The distance of this particle from P is $AP=z=\sqrt{a^2+r^2}$ .The gravitational field at P is dm is along $\overrightarrow{PA}$ and its magnitude is $dE=\frac{Gdm}{z^2}$

The component along PO is $dE\cos\alpha=\frac{Gdm}{z^2}\cos\alpha$

The net gravitational field at P due to the ring is

$$E=\int\frac{Gdm}{z^2}\cos\alpha=\frac{G\cos\alpha}{z^2}\int dm=\frac{GM\cos\alpha}{z^2}=\frac{GMr}{(a^2+r^2)^{3/2}}$$

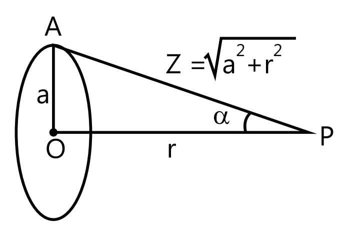


Figure 10.6

The field is directed towards the center of the ring.

(c) Due to uniform disc of mass M and radius a on its axis.

Let us draw a circle of radius x with the center at O. We draw another concentric circle of radius x+dx. The part of the disc enclosed between these two circles can be treated as a uniform ring of radius x. The point P is on its axis at a distance r from the center. The area of this ring is $2\pi x dx$ .The area of the whole disc is πa^2 . As the disc is uniform, the mass of this ring is

$$dm=\frac{M}{\pi a^2}2\pi x dx=\frac{2Mxdx}{a^2}$$

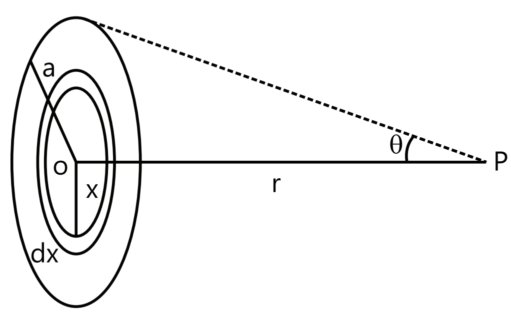


Figure 10.7

The gravitational field at P due to the ring is, by equation,

$$dE=\frac{G\left(\frac{2Mxdx}{a^2}\right)r}{\left(r^2+x^2\right)^{3/2}}=\frac{2GMr}{a^2}\frac{xdx}{\left(r^2+x^2\right)^{3/2}}$$

As x varies from 0 to a, the rings cover up the whole disc. The field due to each of these is in the same direction PO. Thus, the net field due to the whole disc is along PO and its magnitude is

$$E = \int_{o}^{a} \frac{2GMr}{a^2} \frac{xdx}{(r^2+x^2)^{3/2}} = \frac{2GMr}{a^2} \int_{0}^{a} \frac{xdx}{(r^2+x^2)^{3/2}} \qquad ...(i)$$

Let $r^2 + x^2 = z^2$ then 2x dx=2z dz and

$$\int \frac{xdx}{(r^2+x^2)^{3/2}} = \int \frac{zdz}{z^3} = \int \frac{1}{z^2} dz = -\frac{1}{z} = -\frac{1}{\sqrt{r^2-x^2}}$$

From (i) $E = \frac{2GMr}{a^2}\left[-\frac{1}{\sqrt{r^2+x^2}}\right]_0^a = \frac{2GMr}{a^2}\left[\frac{1}{r} - \frac{1}{\sqrt{r^2+a^2}}\right]$

Equation may be expressed in terms of the angle θ subtended by a radius of the disc at P as,

$$E = \frac{2GM}{a^2}(1-\cos\theta).$$

(d) Due to uniform thin spherical shell of mass M and radius a from the triangle OAP,

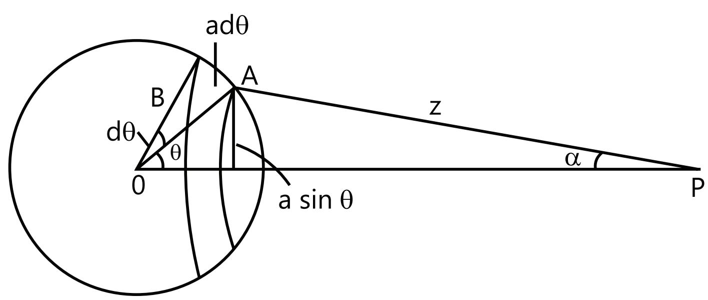


Figure 10.8

$z^2 = a^2 + r^2 - 2ar\cos\theta$ or

$2z\,dz = 2ar\sin\theta d\theta$

or $\sin\theta d\theta = \frac{zdz}{ar}$. ...(ii)

Also from the triangle OAP,

$a^2 = z^2 + r^2 - 2zr\cos\alpha$ or $\cos\alpha = \frac{z^2+r^2-a^2}{2zr}$. ... (iii)

Putting from (ii) and (iii) in (i), $dE = \frac{GM}{4ar^2}\left(1 - \frac{a^2-r^2}{z^2}\right)dz$ or $\int dE = \frac{GM}{4ar^2}\left[z + \frac{a^2-r^2}{z}\right]$

Case I: P is outside the shell (r > a)

In this case, z varies from r – a to r + a. The field due to the whole shell is

$$E = \frac{GM}{4ar^2}\left[z + \frac{a^2-r^2}{z}\right]_{r-a}^{r+a} = \frac{GM}{r^2}$$

We see that the shell may be treated as a point particle of the same mass placed at its center to calculate the gravitational field at an external point.

Case II: P is inside the shell

In this case, z varies from a - r to a + r. The field at P due to the whole shell is $E = \frac{GM}{4ar^2}\left[z + \frac{a^2-r^2}{z}\right]_{a-r}^{a+r} = 0$

Hence the field inside a uniform spherical shell is zero.

(e) Due to uniform solid sphere of mass M and radius a

(i) At an external point r (>a): Let us divide the sphere into thin spherical shells each centered at O. Let the mass of one such shell be dm. To calculate the gravitational field at P, we can replace the shell by a single particle of mass dm placed at the shell that is at O.

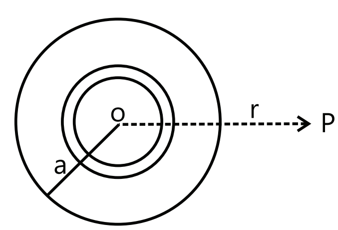


Figure 10.9

The field at P due to this shell is then $dE = \frac{Gdm}{r^2}$

Towards PO. The field due to the whole sphere may be obtained by summing the fields of all the shells making the solid sphere.

Thus, $E = \int dE = \int \frac{Gdm}{r^2} = \frac{G}{r^2}\int dm = \frac{GM}{r^2}$

Thus, a uniform sphere may be treated as a single particle of equal mass placed at its center for calculating the gravitational field at an external point.

(ii) At an internal point r (<a):

Suppose the point P is inside the solid sphere (See Fig 10.10). In this case r<a. The sphere may be divided into thin spherical shells all centered at O.

Suppose the mass of such a shell is dm. If the radius of the shell is less than r, the point is outside the shell. The field due to the shell is $dE = \frac{Gdm}{r^2}$ along PO.

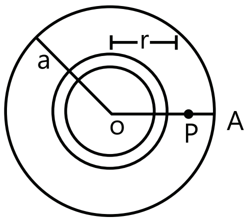


Figure 10.10

If the radius of the shell considered is greater than r, the point P is internal and the field due to such a shell is zero. The total field due to the whole sphere is obtained by summing the fields due to all the shells. As all these fields are along the same direction, the net field is

$$E = \int dE = \int \frac{GdM}{r^2} = \frac{G}{r^2}\int dm \qquad \text{... (i)}$$

Only the masses of the shells with radii less than r should be added to get $z=\sqrt{a^2+r^2}$. These shells form a solid sphere of radius r. The volume of this sphere is $\frac{4}{3}\pi r^3$.The volume of the whole sphere is $\frac{4}{3}\pi a^3$. As the given sphere is uniform, the mass of the sphere of radius r is $\frac{M}{\frac{4}{3}\pi a^3}\left(\frac{4}{3}\pi r^3\right) = \frac{Mr^3}{a^3}$

Thus, $\int dm = \frac{Mr^3}{a^3}$ and by (i) $E = \frac{G}{r^2}\frac{Mr^3}{a^3} = \frac{GM}{a^3}r.$

The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the center of the sphere.

MASTERJEE CONCEPTS

One could assume the whole mass is concentrated at the center of mass (now assume it as point mass) for calculating the gravitation field at an external point for spherical shell, sphere nevertheless of mass distribution (uniformly/non-uniformly)

Mass distribution should be a function of radial distance only.

Remember the Gauss theorem in Electricity?

Equivalent Gauss theorem for gravitational field is $\oint \vec{E}.d\vec{S} = -4\pi G(m)$, m=enclosed mass I guess now you could deduce the note above. Can you?

Nivvedan (JEE 2009, AIR 113)

Illustration 4: Three concentric shells of homogenous mass distribution of masses M_1, M_2 and M_3 having radii a, b and c respectively are situated as shown in Fig. 10.11. Find the force on a particle of mass m **(JEE MAIN)**

(a) When the particle is located at Q.

(b) When the particle is located at P.

Sol: For a particle of mass m, lying at a distance r from the center of the spherical shell of mass M and radius r, the gravitational force of attraction is $\left(\frac{GMm}{r^2}\right)$. If the particle is lying inside the spherical shell then the force of gravitation on it is zero.

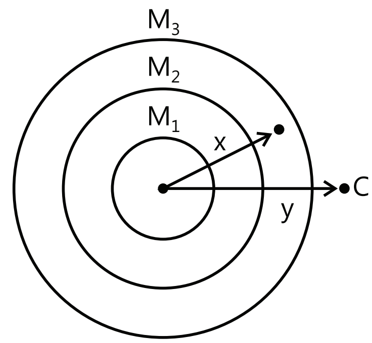


Figure 10.11

Attraction at an external point due to spherical shell of mass M is $\left(\frac{GMm}{r^2}\right)$ while at an internal point is zero.

(a) Point is external to shell M_1, M_2 and M_3,

So, force at Q will be $F_q = \frac{GM_1m}{y^2} + \frac{GM_2m}{y^2} + \frac{GM_3m}{y^2} = \frac{Gm}{y^2}(M_1+M_2+M_3)$

(b) Force at P will be $F_p = \frac{GM_1m}{x^2} + \frac{GM_2m}{x^2} + 0 = \frac{Gm}{x^2}(M_1+M_2)$

Illustration 5: A uniform ring of mass m and radius a is placed directly above a uniform sphere of mass M and of equal radius. The center of the ring is at a distance $\sqrt{3}a$ from the center of the sphere. Find the gravitational force exerted by the sphere on the ring. **(JEE ADVANCED)**

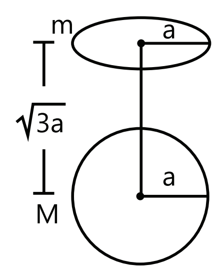


Figure 10.12

Sol: The field due to ring at the center of the sphere can be found easily, as the center of the sphere is lying at the axis of the ring. From Newton's third law of motion the force on the sphere due to the ring will be equal in magnitude to the force exerted by the sphere on the ring.

The gravitational field at any point on the ring due to the sphere is equal to the field due to a single particle of mass M placed at the center of the sphere. Thus, the force on the ring due to the sphere is also equal to the force on it by a particle of mass M placed at this point. By Newton's third law, it is equal to the force on the particle by the ring.

Now the gravitational field due to the ring at a distance $d = \sqrt{3}a$ on its axis is

$$E = \frac{G\,md}{(a^2+d^2)^{3/2}} = \frac{\sqrt{3}\,Gm}{8a^2}$$

The force on a particle of mass M placed here is F=$ME = \frac{\sqrt{3}\,GMm}{8a^2}$. Thus we have used the formula for field due to a ring.

This is also the force due to the sphere on the ring.

5. EARTH'S GRAVITATIONAL FIELD

We have seen what gravitational field is and how an object would interact with other objects. Earth is no different as it creates a gravitational field and interacts with us.

$g = F/m$ (g should be written as g bar and F as F bar. Take care of that)

6. VARIATION IN THE VALUE OF ACCELERATION DUE TO GRAVITY (g)

Variation in the value of g: The value of g varies from place to place on the surface of earth. It also varies as we go above or below the surface of the earth. Thus, value of g depends on the following factors:-

(a) Shape of the earth: The earth is not a perfect sphere. It is somewhat flat at the two poles. The equatorial radius is approximately 21 km more than the polar radius. And since

$$g = \frac{GM}{R^2} \quad \text{Or} \quad g \propto \frac{1}{R^2}$$

The value of g is minimum at the equator and maximum at the poles.

(b) Height above the surface of the earth: The gravitational force on mass m due to Earth of mass M at height h above the surface of earth is

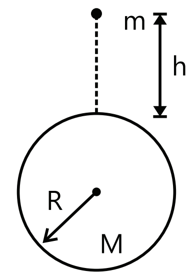


Figure 10.13

$$F = \frac{GMm}{(R+h)^2}$$

So the acceleration due to gravity is $g' = \frac{F}{m} = \frac{GM}{(R+h)^2}$

This can also be written as, $g' = \dfrac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$ Or $g' = \dfrac{g}{\left(1+\frac{h}{R}\right)^2}$ as $\frac{GM}{R^2} = g$

Thus, $g' < g$ i.e., the value of acceleration due to gravity g goes on decreasing as we go above the surface of earth. Further,

$$g' = g\left(1+\frac{h}{R}\right)^{-2} \quad \text{or} \quad g' \approx g\left(1-\frac{2h}{R}\right) \text{ if } h << R$$

So on going above the surface of the earth, acceleration due to gravity decreases. Note that mass is always constant.

(c) Depth below the surface of the earth: Let an object of mass m is situated at a depth h below the earth's surface. Its distance from the center of earth is (R - h). This mass is situated at the surface of the inner solid sphere and lies inside the outer spherical shell. The gravitational force of attraction on a mass inside a spherical shell is always zero. Therefore, the object experiences gravitational attraction only due to inner solid sphere.

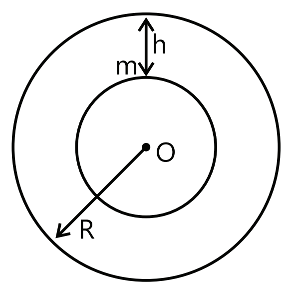


Figure 10.14

The mass of this sphere is $M' = \left(\frac{M}{4/3\pi R^3}\right)\frac{4}{3}\pi(R-h)^3$ or $M' = \frac{(R-h)^3}{R^3}M$

$$F = \frac{GM'm}{(R-h)^2} = \frac{GMm(R-h)}{R^3} \quad \text{and} \quad g' = \frac{F}{m}$$

Substituting the values, we get $g' = g\left(1-\frac{h}{R}\right)$ i.e., $g' < g$

(d) Axial rotation of the earth: Let us consider a particle P at rest on the surface of the earth, in latitude ϕ. Then the pseudo force acting on the particles is $mr\omega^2$ in outward direction. The true acceleration g is acting towards the center O of the earth. Thus, the effective accelerating g' is the resultant of g and $r\omega^2$ or

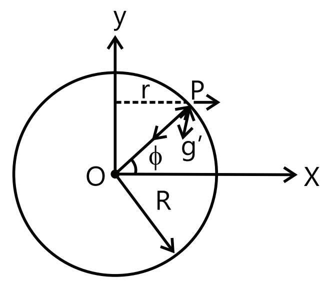


Figure 10.15

$$g' = \sqrt{g^2 + (r\omega^2)^2 + 2g(r\omega^2)\cos(180-\phi)}$$

or $$g' = \sqrt{g^2 + r^2\omega^4 - 2gr\omega^2\cos\phi} \quad \text{... (i)}$$

Here, the term $r^2\omega^4$ comes out to be too small as $\omega = \frac{2\pi}{T} = \frac{2\pi}{24\times 3600}$

rad/s is small. Hence, this term can be ignored. Also, $r = R\cos\phi$. Therefore, Eq. (i) can be written as $g' = (g^2 - 2gR\omega^2\cos^2\phi)^{1/2}$

$$= g\left(1 - \frac{2R\omega^2\cos^2\phi}{g}\right)^{1/2} = g\left(1 - \frac{R\omega^2\cos^2\phi}{g}\right)$$

Thus, $g' = g - R\omega^2\cos^2\phi$ $\quad R\omega^2$ is almost 0.03 m/s^2

MASTERJEE CONCEPTS

They is always a decrease in the value of acceleration due to gravity from that of g at the surface irrespective of the condition.

If earth were to rotate faster 'g' would decrease at all points except at the poles. Guessed it? ϕ is 90 at poles. Also remember ϕ is 0 at equator.

Chinmay S Purandare (JEE 2012, AIR 698)

Illustration 6: Suppose the earth increases its speed of rotation. At what new time period will the weight of a body on the equator become zero? Take g = 10 m/s^2 and radius of earth R = 6400km. **(JEE MAIN)**

Sol: When rotational speed of earth is increased, the centrifugal force acting on the particle at rest at equator also increases. At the equator, the centrifugal force is opposite to the force of gravity. Thus the apparent value of g is $g' = g - R\omega^2$. For mass of body to be zero at the equator, g' = 0 i.e. $\omega = \sqrt{\frac{g}{R}}$. The time period of rotation is $T = \frac{2\pi}{\omega}$.

The weight will become zero, when $g' = 0$ or $g - R\omega^2 = 0$ (on the equator $g' = g - R\omega^2$)

or $\omega = \sqrt{\frac{g}{R}}$; $\therefore \frac{2\pi}{T} = \sqrt{\frac{g}{R}}$ or $T = 2\pi\sqrt{\frac{R}{g}}$

Substituting the values, $T = \frac{2\pi\sqrt{\frac{6400\times10^3}{10}}}{3600}h$ or $T = 1.4$ h

Thus, the new time period should be 1.4 h instead of 24 h for the weight of a body to be zero on the equator.

Illustration 7: A simple pendulum has a time period exactly 2 s when used in a laboratory at North Pole. What will be the time period if the same pendulum is used in a laboratory at equator? Account for the earth's rotation only. Take $g = \frac{GM}{R^2} = 9.8\,m/s^2$ and radius of earth=6400 km. **(JEE ADVANCED)**

Sol: The time period of simple pendulum is given by $t = 2\pi\sqrt{\frac{\ell}{g}}$ where ℓ is the length of pendulum. At the equator value of acceleration due to gravity 'g' is different than at the pole. The apparent value of g is $g' = g - R\omega^2$. Thus the time periods will be different.

Consider the pendulum in its mean position at the North Pole. As the pole is on the axis of rotation, the bob is in equilibrium. Hence in the mean position, the tension T is balanced by earth's attraction. Thus, $T = \frac{GMm}{R^2} = mg$.

The time period t is $t = 2\pi\sqrt{\frac{\ell}{T/m}} = 2\pi\sqrt{\frac{\ell}{g}}$ ... (i)

At equator, the lab and the pendulum rotate with the earth at angular velocity $\omega = \frac{2\pi \text{ radian}}{24 \text{ hour}}$ in a circle of radius equal to 6400 km. Using Newton's second law,

$$\frac{GMm}{R^2} - T' = \omega^2 R \text{ or, } T' = m(g - \omega^2 R)$$

Where T' is the tension in the string.

The time period will be

$$t' = 2\pi\sqrt{\frac{l}{(T'/m)}} = 2\pi\sqrt{\frac{l}{g-\omega^2 R}} \quad \text{... (ii)}$$

By (i) and (ii)

$$\frac{t'}{t} = \sqrt{\frac{g}{g-\omega^2 R}} = \left(1 - \frac{\omega^2 R}{g}\right)^{-1/2} \quad \text{or,} \quad t' = t\left(1 + \frac{\omega^2 R}{2g}\right)$$

Putting the values, $t' = 2.004$ seconds.

7. GRAVITATIONAL POTENTIAL ENERGY

Suppose I would like to move a particle form another particle's field, work is either done against the gravitational field or extracted from it. This negative work is called as Gravitational Potential energy.

Gravitational force is a conservative in nature. Work done by gravitational field= $U_f - U_i = -\int_i^f \vec{F}.\vec{dr}.$

Let a particle of mass m_1 be kept fixed at a point A (See Fig 10.16) and another particle of mass m_2 is taken from a point B to a point C. Initially, the distance between the particles is $AB = r_1$ and finally it becomes $AC = r_2$. We have to calculate the change in potential energy of the system of the two particles as the distance changes from r_1 to r_2.

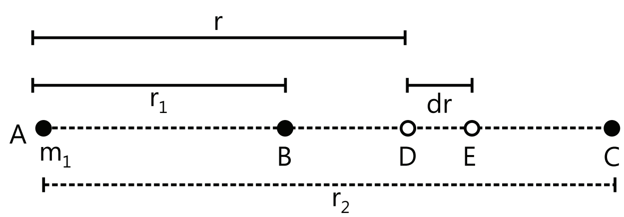


Figure 10.16

Consider a small displacement when the distance between the particles changes from r to r + dr. In the Fig 10.16, this corresponds to the second particle going from D to E.

The force on the second particle is $F = \frac{Gm_1m_2}{r^2}$ along $\overrightarrow{DA}$

The work done by the gravitational force in the displacement is $dW = -\frac{Gm_1m_2}{r^2}dr.$

The change in potential energy of the two-particle system during this displacement is $dU = -dW = \frac{Gm_1m_2}{r^2}dr.$

The change in potential energy as the distance between the particles from r_1 to r_2 is

$$U(r_2) - U(r_1) = \int dU = \int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2}dr = Gm_1m_2\int_{r_1}^{r_2}\frac{1}{r^2}dr = Gm_1m_2\left[-\frac{1}{r}\right]_{r_1}^{r_2} = Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

This is the change in potential energy of the particles when moved from B to C.

Suppose the same particles which are of mass m_1 and m_2 are very far from each other and we need to calculate the change in potential energy when the distance between them becomes r. Then using above formulae,

we get $$U(r) - U(\infty) = Gm_1m_2\left[\frac{1}{\infty} - \frac{1}{r}\right] = -\frac{Gm_1m_2}{r}$$

We make a standard assumption that the potential energy of the two-particle system to be zero when the distance between them is infinity. This means that we choose $U(\infty) = 0$.

Note: Just as one assumed current to be in opposite direction with the flow of electrons, the potential at infinity is assumed to be zero.

8. GRAVITATIONAL POTENTIAL

The potential at a point may also be defined as the work done per unit mass by an external agent in bringing a particle slowly from the reference point to the given point. Generally the reference point is chosen at infinity so that the potential at infinity is zero.

MASTERJEE CONCEPTS

By slowly I mean, the particle is moved in such a way that there is no increase in Kinetic energy.

Since the Kinetic energy of the particle is zero, from the work energy theorem, the total work done is change in potential energy. So, what is the difference between the Potential and Potential energy? Observe it is the work done per unit mass.

Nitin Chandrol (JEE 2012, AIR 134)

We define the "change in potential" $V_B - V_A$ between the two points as $V_B - V_A = \frac{U_B - U_A}{m}$

Calculation of some Gravitational potentials:

(a) Potential due to point mass M at a point P which is at a distance r

(b) (ii) Potential due to Uniform ring of radius "a" and mass M at a point P on its axis.

(c) $V_{(r)} = \frac{U_{(r)} - U_{(\infty)}}{m}$

But $U(r) - U(\infty) = -\frac{GMm}{r}$ so that $V = -\frac{GM}{r}$

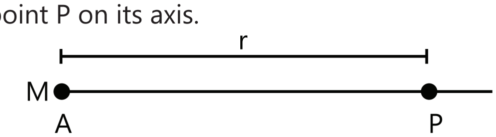


Figure 10.17

The gravitational potential due to a point mass M at a distance r is $-\frac{GM}{r}$

(d) Consider any small part of the ring of mass dm. The point P is at a distance $z = \sqrt{a^2 + r^2}$ from dm.

$$dV = -\frac{GdM}{r} = -\frac{Gdm}{\sqrt{a^2 + r^2}};$$

$$V = \int dV = \int -\frac{Gdm}{\sqrt{a^2 + r^2}} = -\frac{G}{\sqrt{a^2 + r^2}} \int dm = -\frac{GM}{\sqrt{a^2 + r^2}}$$

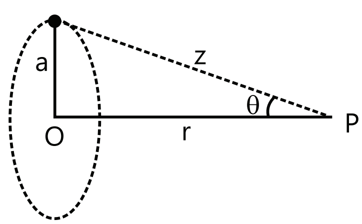


Figure 10.18

MASTERJEE CONCEPTS

Remember that potential is a scalar quantity and one can directly add the contributions due to each of the point masses.

Potential due to Uniform Thin spherical shell and due to Uniform sphere can be derived similarly and here is the table of all the results.

MASTERJEE CONCEPTS

	Potential	Gravitational Field
Point Mass at a distance r	$\frac{-GM}{r}$	$\frac{-GM}{r^2}\vec{e}_r$
Uniform Ring at a point on its axis	$\frac{-GM}{\sqrt{a^2+r^2}}$	$\frac{GMr}{\left(a^2+r^2\right)^{3/2}}$ towards center of ring
Uniform Thin spherical shell	$\frac{-GM}{a}$ (inside) $\frac{-GM}{r}$ (outside)	0 (inside) $\frac{GM}{r^2}$ (outside)
Uniform Solid Sphere	$\frac{-GMr^2}{a^3}$ (Inside) $\frac{-GM}{2a^3}\left(3a^2-r^2\right)$ (outside)	$\frac{GMr}{a^3}$ (inside) $\frac{GM}{r^2}$ (outside)

Only the magnitudes of gravitational field are written. As the gravitational force is attractive in nature, the direction could be easily found out.

Gravitational force, potential and potential energy all are taken with negative sign because the gravitational force is always attractive in nature.

$E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y}$ and $E_z = -\frac{\partial V}{\partial z}$

Potential using the field for various cases $V\left(\vec{r}_2\right) - V\left(\vec{r}_1\right) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}.\vec{dr}$.

B Rajiv Reddy (JEE 2012, AIR 11)

Illustration 8: A particle of mass 1 kg is kept on the surface of a uniform sphere of mass 20 kg and radius 1.0 m. Find the work to be done against the gravitational force between them to take the particle away from the sphere. **(JEE MAIN)**

Sol: The work done in moving a particle away from the sphere will be equal to the change in gravitational potential energy of the particle in the gravitational field of the sphere.

Potential at the surface of sphere, $V = -\frac{GM}{R} = -\frac{(6.67\times10^{-11})(20)}{1} \text{J/kg} = -1.334\times10^{-9}\ \text{J/kg}$

i.e., 1.334×10^{-9} J work is obtained to bring a mass of 1 kg from infinity to the surface of sphere. Hence, the same amount of work will have to be done to take the particle away from the surface of sphere. Thus, $W = 1.334 \times 10^{-9}$ J

Illustration 9: A particle is fired vertically upward with a speed of 9.8 km/s. Find the maximum height attained by the particle. Radius of earth = 6400 km and g at the surface=9.8 m/s^2. Consider only earth's gravitation.

(JEE MAIN)

Sol: Particle initially moves with kinetic energy only in upwards direction opposite to the gravitation pull of earth. The loss in its kinetic energy is equal to the gain in the potential energy. At the highest point of its vertical motion, kinetic energy is converted completely into potential energy.

At the surface of the earth, the potential energy of the earth-particle system is $-\frac{GMm}{R}$ with usual symbols. The kinetic energy is $\frac{1}{2}mv_0^2$ where v_0 = 9.8 km/s. At the maximum height the kinetic energy is zero. If the maximum height reached is H, the potential energy of the earth-particle system at this instant is $-\frac{GMm}{R+H}$. Using conservation of energy, $-\frac{GMm}{R} + \frac{1}{2}mv_0^2 = -\frac{GMm}{R+H}$

Writing $GM = gR^2$ and dividing by m, $-gR + \frac{v_0^2}{2} = \frac{-gR^2}{R+H}$ or $\frac{R^2}{R+H} = R - \frac{v_0^2}{2g}$ or $R + H = \frac{R^2}{R - \frac{v_0^2}{2g}}$ Putting the values of R, v_0 and g on the right side,

$$R + H = \frac{(6400km)^2}{6400km - \frac{(9.8kms^{-1})^2}{2 \times 9.8ms^{-2}}} = \frac{(6400km)^2}{1500km} = 27300km \text{ or } H = (27300 - 6400)km = 20900km.$$

Illustration 10: Two particles of equal masses go round a circle of radius R under the action of their mutual gravitational attraction. Find the speed of each particle. **(JEE MAIN)**

Sol: As the particles go around the circle they always remain diametrically opposite to each other. To sustain their respective circular motion the necessary centripetal acceleration is provided by the gravitation force of attraction between them.

The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius. Consider the motion of one of the particles. The force on the particle is $F = \frac{Gm^2}{4R^2}$. If Thus, by Newton's law,

$$\frac{Gm^2}{4R^2} = \frac{mv^2}{R} \text{ or } v = \sqrt{\frac{Gm}{4R}}$$

9. BINDING ENERGY

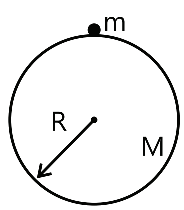


Figure 10.19

It is the energy due to which a system is bound. Suppose the mass m is placed on the surface of earth. The radius of the earth is R and its mass M. Then, the kinetic energy of the particle K=0 and potential energy of the particle is $U = -\frac{GMm}{R}$.

Therefore, the total mechanical energy of the particle is, $E = K + U = 0 - \frac{GMm}{R}$ or $E = -\frac{GMm}{R}$

It is due to this energy, the particle is attached to the earth. If this amount of energy is supplied to the particle in any form (normally kinetic), the particle no longer remains bound to the earth. It goes out of the gravitational field of earth.

Illustration 11: Assuming the earth to be a sphere of uniform mass destiny, calculate the energy needed to completely disassemble it against the gravitational pull amongst its constituent particles. Given the product of mass and radius of the earth $= 2.5\times10^{31}$ kgm, $g = 10\,m/s^2$. **(JEE MAIN)**

Sol: The work done to completely disassemble the earth will be equal to change in potential energy of the earth. Initial potential energy is negative and final will be zero.

If M and R are the mass and radius of the earth, then the density ρ of the earth is $\rho = \frac{3M}{4\pi R^3}$

The earth may be supposed to be made up of a large number of thin concentric spherical shells. It can be disassembled by removing such shells one by one. When a sphere of radius x is left, the energy needed to remove a shell of thickness lying between x and $x + dx$ is $dU = \frac{Gm_1m_2}{x}$

Where m_1 = mass of the sphere of radius $x = \frac{4}{3}\pi x^3 p$,

and m_2 = mass of the spherical shell of radius x and thickness dx $= 4\pi x^2 dx\rho$

$$\therefore \quad dw = dU = \frac{G\left(\frac{4}{3}\pi x^3\rho\right)\left(4\pi x^2 dx\rho\right)}{x} = \frac{16}{3}G\pi^2\rho^2x^4dx$$

$$\text{Total energy required } U = \int dU = \frac{16G\pi^2\rho^2}{3}\int_0^R x^4dx = \frac{16G\pi^2\rho^2}{3}\frac{R^5}{5} = \frac{16}{15}G\pi^2\left(\frac{M}{(4/3)\pi R^3}\right)^2 R^5 = \frac{3}{5}\frac{GM^2}{R}$$

$$= \frac{3}{5}gMR = \frac{3}{5}\times10\times2.5\times10^{31} = 1.5\times10^{32}\text{ J.}$$

10. ESCAPE VELOCITY

The minimum velocity needed to take a particle infinitely away from the earth is called the escape velocity. On the surface of earth its value 11.2 km/s.

As we discussed the binding energy of a particle on the surface of earth kept at rest is $\frac{GMm}{R}$. If this much energy in the form of kinetic energy is supplied to the particle, it leaves the gravitational field of the earth. So, if v_e is the escape velocity of the particle, then

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R} \quad \text{or} \quad v_e = \sqrt{\frac{2GM}{R}} \quad \text{or } v_e = \sqrt{2gR} \qquad \text{as} \quad g = \frac{GM}{R^2}$$

MASTERJEE CONCEPTS

Escape velocity is independent of angle of projection.

Anand K (JEE 2011, AIR 47)

Illustration 12: Calculate the escape velocity from the surface of moon. The mass of the moon is 7.4×10^{22} kg and radius $= 1.74\times10^6$ m **(JEE MAIN)**

Sol: Escape velocity of any object placed on moon is given by $v_e = \sqrt{\frac{2GM_m}{R_m}}$

Escape velocity from the surface of moon is $v_e = \sqrt{\frac{2GM_m}{R_m}}$

Substituting the values, we have $v_e = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.4 \times 10^{22}}{1.74 \times 10^{6}}} = 2.4 \times 10^{3}$ m/s or2.4 km/s

11. SATELLITES

Satellites are generally of two types:

Natural Satellites: Moon is a natural satellite of the earth.

Artificial Satellite: These are launched in to space by humans and they help us in weather forecasting, telecommunications etc. The path of these satellites is elliptical with the center of earth at a focus.

Orbital Speed: The necessary centripetal force to the satellite is being provided by the gravitational force exerted by the earth on the satellite. Thus,

$\therefore \quad v_o = \sqrt{\frac{GM}{r}} \quad$ or $\quad v_o \propto \frac{1}{\sqrt{r}}$

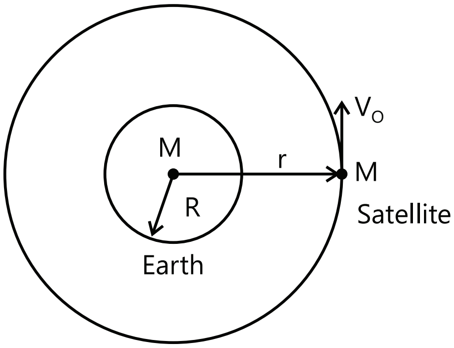


Figure 10.20

Hence, the orbital speed (v_o) of the satellite decreases as the orbital radius (r) of the satellite increases. Further, the orbital speed of a satellite close to the earth's surface $(r \approx R)$ is, $v_o = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \frac{v_e}{\sqrt{2}}$;

Substituting $\quad v_e = 11.2\text{km/s}; \; v_o = 7.9\text{km/s}$

Period of Revolution: The period of revolution (T) is given by $T = \frac{2\pi r}{v_o}$ or $T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$ or $\quad T = 2\pi\sqrt{\frac{r^3}{GM}}$

Or $T = 2\pi\sqrt{\frac{r^3}{gR^2}} \qquad$ (as $GM = gR^2$)

Energy of Satellite: The potential energy of the system is $\quad U = -\frac{GMm}{r}$

The kinetic energy of the satellite is, $K = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{GM}{r}\right)$

or $\quad K = \frac{1}{2}\frac{GMm}{r}$

The total energy is, $\quad E = K + U = -\frac{GMm}{2r}$ or $\quad E = -\frac{GMm}{2r}$

This energy is constant and negative, i.e., the system is closed. The farther the satellite from the earth the greater its total energy.

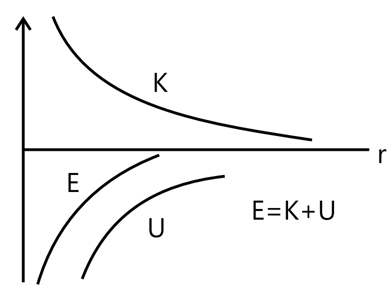


Figure 10.21

MASTERJEE CONCEPTS

The velocity of a satellite is independent of its mass. It only depends upon the mass of the planet around which it revolves.

What if the time period of rotation of satellite is exactly 24 hours just as the time period of rotation of earth? Its position w.r.t earth is fixed right! Try calculating the distance from the earth's surface. By the way, these satellites are called Geo-stationary (stationary w.r.t earth) satellites.

Yashwanth Sandupatla (JEE 2012, AIR 821)

Illustration 13: Consider an earth's satellite so positioned that it appears stationary to an observer on earth and serves the purpose of a fixed relay station for international transmission of TV and other communications. What would be the height at which the satellite should be positioned and what would be the direction of its motion? Given that the radius of the earth is 6400 km and acceleration due to gravity on the surface of the earth is 9.8 m/s^2. **(JEE ADVANCED)**

Sol: For any artificial satellite to appear stationary with respect to a point on earth, it must rotate with the same angular speed as that of the earth and in the direction of motion as of the earth. The angular velocity of the satellite at height h above earth surface is given by $\omega = \sqrt{GM/r^3}$ where r=R+h.

For a satellite to remain above a given point on the earth's surface, it must rotate with the same angular velocity as the point on earth's surface. Therefore the satellite must rotate in the equatorial plane from west to east with a time period of 24 hours.

Now as for a satellite orbital velocity is $v_0 = \sqrt{GM/r}$

$$T = \frac{2\pi r}{v_o} = 2\pi r\sqrt{\frac{r}{GM}} = 2\pi r\sqrt{\frac{r}{gR^2}} \text{ (as } g = GM/R^2 \text{) or } r = \left[gR^2 \frac{\frac{1}{3}}{4\pi^2} \right] = 4.23\times107\text{m} = 42300\text{km}$$

So the height of the satellite above the surface of earth, $h = r - R = 42300 - 6400 \approx 36000\text{km}$

[The speed of a geostationary satellite $v_o = R\sqrt{g/r} = r\omega = 3.1\,\text{km/s}$]

Illustration 14: Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 h and 8 h respectively. The radius of the orbit of S_1 is 10^4 km. When S_2 is closest to S_1 find (a) the speed of S_2 relative to S_1 and (b) the angular speed of S_2 as observed by an astronaut in S_1.

(JEE ADVANCED)

Sol: According to Kepler's laws of planetary motion, $T^2 \propto R^3$. The orbital velocity of the satellite $v_0 = \frac{2\pi R}{T} = R\omega$ where ω is the angular velocity of revolution of satellite.

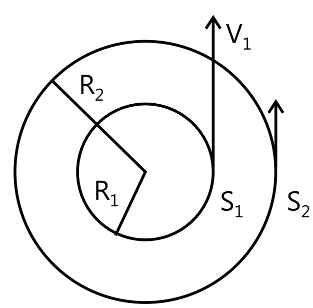


Figure 10.22

Let the mass of the planet be M, that of S_1 be m_1 and of S_2 be m_2.

Let the radius of the orbit of S_1 be $R_1 (= 10^4\text{km})$ and so S_2 be R_2.

Let v_1 and v_2 be the linear speeds of S_1 and S_2 with respect to the planet. The given Fig 10.22 shows the situation.

As the square of the time period is proportional to the cube of the radius,

$$\left(\frac{R_2}{R_1}\right)^3 = \left(\frac{T_2}{T_1}\right)^2 = \left(\frac{8h}{1h}\right)^2 = 64 \quad \text{or} \quad \frac{R_2}{R_1} = 4 \text{ or } R_2 = 4R_1 = 4\times10^4\text{km}$$

Now the time period of S_1 is 1 h.

So, $\frac{2\pi R_1}{v_1} = 1h$ or $\quad v_1 = \frac{2\pi R_1}{1h} = 2\pi \times 10^4 kmh^{-1}$

Similarly, $\quad v_2 = \frac{2\pi R_2}{8h} = \pi \times 10^4 kmh^{-1}$

(a) At the closest separation, they are moving in the same direction. Hence the speed of S_2 with respect to S_1 is $|v_2 - v_1| = \pi \times 10^4 kmh^{-1}$

(b) As seen from S_1, the satellite S_2 is at a distance $R_2 - R_1 = 3\times 10^4 km$ at the closest separation. Also, it is moving at $\pi \times 10^4 kmh^{-1}$ in a direction perpendicular to the line joining them.

Thus, the angular speed of S_2 as observed by S_1 is $\omega = \frac{\pi \times 10^4 kmh^{-1}}{3\times 10^4 km} = \frac{\pi}{3} radh^{-1}$

Illustration 15: A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity is now to be added to the spaceship in the orbit to overcome the gravitational pull? Radius of earth $= 6400$ km, $g = 9.8\ m/s^2$. **(JEE MAIN)**

Sol: The potential energy of the spaceship close to the earth is negative (- mgR). The orbital speed close to the earth is $v = \sqrt{gR}$, so the kinetic energy is mgR/2. The total energy is - mgR/2. We need to provide the additional kinetic energy = mgR/2 such that the spaceship escapes the gravitational pull of the earth.

The extra kinetic energy to be given is $\frac{mv^2}{2} = \frac{mgR}{2}$, so that the extra velocity given is $v' = \sqrt{gR}$.

The velocity is $v' = \sqrt{9.8 \times 6400000} = 7.91 \times 10^3\ m/s = 7.91\ km/s$

Illustration 16: An artificial satellite is moving in a circular orbit around the earth with a speed equal to one fourth the magnitude of escape velocity from the earth.

(i) Determine the height of the satellite above the earth's surface.

(ii) If the satellite is stopped suddenly in its orbit and allowed to fall freely towards the earth, find the speed with which it hits the surface of the earth. **(JEE MAIN)**

Sol: For satellite the escape velocity is $v_e = \sqrt{2Rg}$. According to given data the satellite is moving in the orbit with one fourth the magnitude of this velocity. When satellite stops revolving, it falls freely under action of gravity from the height h above the surface of the earth. The loss in the gravitational potential energy in falling height h is equal to gain in the kinetic energy of the satellite.

(i) Let M and R be the mass and radius of the earth respectively. Let m be the mass of satellite. Here escape velocity from earth $v_e = \sqrt{(2Rg)}$

Velocity of satellite $v_g = \frac{v_e}{4} = \sqrt{(2Rg)}/4$ ...(i)

Further $v_c = \sqrt{\left(\frac{GM}{r}\right)} = \sqrt{\left(\frac{R^2 g}{R+h}\right)} \quad \therefore \quad v_g^2 = \frac{R^2 g}{R+h}$ ...(ii)

From equation (i) and (ii), we get H=7R=44800km

(ii) Now, the total energy at height h=total energy on earth's surface (principle of conservation of energy). Let it reach earth's surface with velocity v.

$\therefore \quad 0 - GM\frac{m}{R+h} = \frac{1}{2}mv^2 - GM\frac{m}{R}$ Or $\frac{1}{2}mv^2 = \frac{GMm}{R} - \frac{GMm}{7R} \quad (\because h = 7R)$

Solving we get $v = \sqrt{12Rg/7}$ $\therefore$ $v = \sqrt{(1.714 \times 6400 \times 10^3 \times 9.8)} = 10.368\,km/sec$

12. PLANETS AND THEIR MOTION

12.1 Law of Orbits

All the planets move in elliptical orbits with the sun as one of its focii.

12.2 Law of Areas

The radius vector from the sun at the focus of elliptical orbit to the planet sweeps out equal areas in equal intervals of time.

If the radius vector R sweeps an angle $d\theta$ in time dt, area ASB swept by radius vector in time $dt = dA = \frac{1}{2} \times R \times Rd\theta$

$\therefore\ dA = \frac{1}{2}R^2\frac{d\theta}{dt}dt = \frac{1}{2}\omega R^2 dt$

Areal velocity $= \frac{\text{area}}{\text{time}} = \frac{dA}{dt} = \frac{1}{2}\omega R^2$

So ωR^2 is constant for area SAB and area SCD. It shows that the angular momentum $mR^2\omega$ is conserved for planetary motion. When R decreases, ω increases so that ωR^2 is constant.

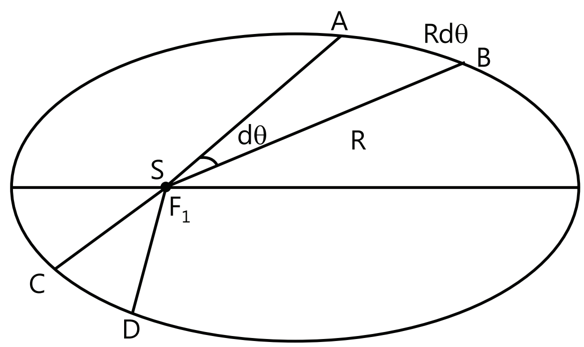


Figure 10.23

12.3 Laws of Periods

The square of the time period of revolution of a planet is proportional to the cube of the mean distance of the planet from the sun.

If a is the mean distance of sun from the planet, T^2 is proportional to a^3 or $T^2 = Ka^3$ where K is a constant.

If a_1 and a_2 are semi-major axis of the orbits of two planets around the sun with respective time periods T_1 and T_2, then $\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$

MASTERJEE CONCEPTS

Observe the time period of rotation of satellite. Got it? (It follows Kepler's third law too)

When the planet is farthest from Sun, it is said to be at the Apogee of Aphelion.

When the planet is at nearest to the Sun, it is said to be at Perigee or Perhilion.

GV Abhinav (JEE 2012, AIR 329)

Illustration 17: The minimum and maximum distance of a satellite from the center of the earth are 2R and 4R respectively, where R is the radius of earth and M is the mass of the earth. Find:

(a) Its minimum and maximum speeds,

(b) Radius of curvature at the point of minimum distance. **(JEE ADVANCED)**

Sol: The speed of the satellite is minimum when is at the maximum distance from the earth and vice versa. At the point of minimum or maximum distance from earth the velocity vector is perpendicular to the radius vector from the earth. Apply law of conservation of angular momentum and energy at the two points.

(a) Applying conservation of angular momentum

$$mv_1(2R) = mv_2(4R) \quad v_1 = 2v_2 \qquad \text{...(i)}$$

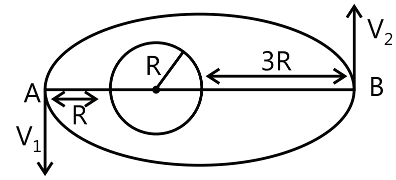


Figure 10.24

From conservation of energy

$$\frac{1}{2}mv_1^2 - \frac{GMm}{2R} = \frac{1}{2}mv_2^2 - \frac{GMm}{4R} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_2 = \sqrt{\frac{GM}{6R}}, \qquad v_1 = \sqrt{\frac{2GM}{3R}}$$

(b) If r is the radius of curvature at point A

$$\frac{mv_1^2}{r} = \frac{GMm}{(2R^2)}; \quad r = \frac{4v_1^2R^2}{GM} = \frac{8R}{3} \qquad \text{(Putting value of } v_1\text{)}$$

Illustration 18: The planet Neptune travels around the Sun with a period of 165 year. Show that the radius of its orbit is approximately thirty times that of Earth's orbit, both being considered as circular. **(JEE ADVANCED)**

Sol: According to the Kepler's laws of planetary motion $T^2 \propto R^3$ where T is the time period of revolution and R is the radius of the orbit of revolution of planet. Taking the ratio of time periods of revolution of Earth and Neptune, we get the ratio of radius of their orbits.

$T_1 = T_{Earth} = 1$ year; $T_2 = T_{Neptune} = 165$ year $= 165\ T_1$

Let R_1 and R_2 be the radii of the circular orbits of Earth and Neptune respectively.

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3} \quad \therefore \quad R_2^3 = \frac{R_1^3T_2^2}{T_1^2} \quad \text{or} \quad R_2^3 = \frac{R_1^3 \times 165^2}{1^2}$$

$$\therefore \quad R_2^3 = 165^2R_1^3 \quad \text{or} \quad R_2 \approx 30R_1$$

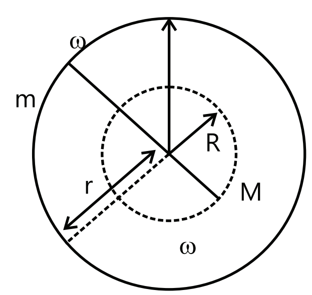
13. MOTION ABOUT THE CENTRE OF MASS

As shown in the Fig 10.25, for the case of circular orbits, two objects are moving about their common center of mass. If we consider the motion of the smaller body,

$$\frac{GMm}{(r+R)^2} = m\omega^2 r$$

The revised law of periods in

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3\left(1+\frac{R}{r}\right)^2$$



Two bodies moving in circular orbits under the influences of each other's gravitational attraction

Figure 10.25

Illustration 19: A pair of stars rotate about their common center of mass. One of them has mass m and the other 2m. Their centers are a distance d apart, d being large compared to the size of either star.

(a) Derive an expression for the period or rotation of the stars about their common center of mass in terms of d, m and G

(b) Compare the angular moments of the two stars about their common center of mass.

(c) Compare the kinetic energies of the two stars. **(JEE MAIN)**

Sol: The gravitational pull between two stars provides the necessary centripetal acceleration to make them revolve in a circular orbit. The time period of revolution of each star is $T=\frac{2\pi}{\omega}$. The angular momentum of the revolving body is given by L=I ω = m $r^2\omega$. And the kinetic energy is given by $E=\frac{I\omega^2}{2}$.

The center of mass O is at a distance 2d/3 from the star of mass m and d/3 from the star of mass 2m. Both the stars rotate with the same angular velocity ω.

(a) Since the gravitational force provides the centripetal force, then

$$m\left(\frac{2d}{3}\right)\omega^2=\frac{Gm.2m}{d^2}\Rightarrow \quad \omega=\sqrt{3Gm/d^3} \quad \text{or} \quad T=\frac{2\pi}{\omega}=2\pi\sqrt{d^3/3Gm}$$

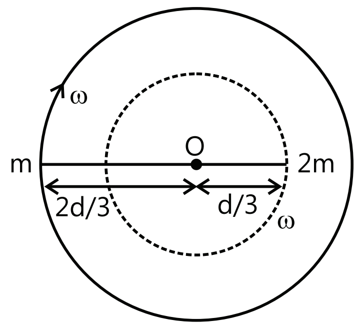


Figure 10.26

(b) Ratio of angular momenta

$$\frac{L_{small}}{L_{large}}=\frac{m(2d/3)^2\,\omega}{2m(d/3)^2\,\omega}=2$$

(c) Ratio of kinetic energies

$$\frac{E_{small}}{E_{large}}=\frac{\frac{1}{2}i_{small}\omega^2}{\frac{1}{2}i_{big}\omega^2}=2$$

PROBLEM-SOLVING TACTICS

1. Most of the problems are easy, as gravitation and electrostatics are analogous to each other. Just be careful that gravitational force is always attractive, whereas electrostatic force can be attractive as well as repulsive and make changes as necessary.

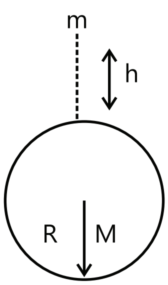
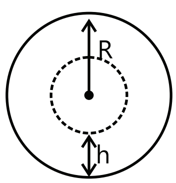
2. Assumptions are appreciated in real cases of satellites and planetary motion.

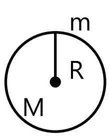
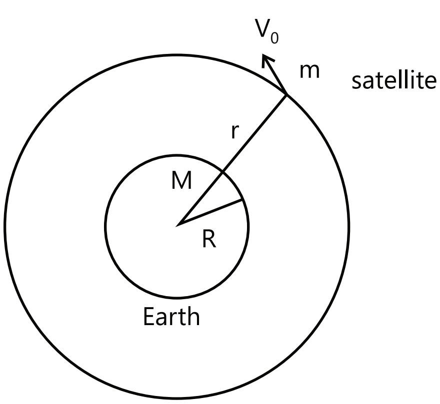
3. Ideas and concepts of circular motion must be strong because they are generally applied here.

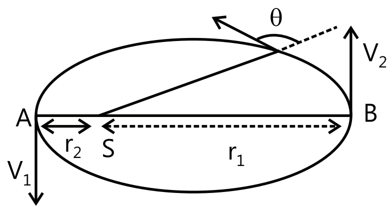
4. While dealing practical cases on Earth, be careful about Earth's rotation on its own axis.

5. Most questions are solved with ease by using work-energy theorem and laws of motion

FORMULAE SHEET

S. No.	Description	Formulae	
1	Magnitude of gravitational force between two particles of mass m_1 & m_2 placed at a distance r is	$F = \frac{Gm_1m_2}{r^2}$ $G = 6.67 \times 10^{-11} N-m^2/kg^2$ Note: It acts along the line joining two particles.	
2	Acceleration due to gravity (g)	$g = \frac{GM}{R^2}$ SI units:- m/s^2 M is the mass of the earth and its radius R.	
3		Gravitational force $= \frac{GMm}{(R+h)^2}$ Acceleration due to gravity $= g' = F/m = \frac{GM}{(R+h)^2}$ If $h << R$ $g' = g\left(1 - \frac{2h}{R}\right)$	
4		At a certain, Depth H, acceleration due to gravity g' is $g' = g\left(1 - \frac{h}{R}\right)$ g is acceleration due to gravity at surface of earth.	
5	Effect of g due to axial rotation of earth	$g' = g - R\omega^2 \cos^2 \phi$ g' is the acceleration due to gravity on the particle on the earth surface in latitude ϕ.	
6	Gravitational field strength	$\vec{E} = \frac{\vec{F}}{m}$ SI unit is N/kg.	
		Gravitational Field	**Gravitational Potential**
7	Point Mass	$\frac{GM}{r^2}$	$-\frac{GM}{r}$
8	Uniform ring at point on its axis	$\frac{GMr}{(a^2+r^2)^{3/2}}$ (towards center of ring)	$-\frac{GM}{\sqrt{a^2+r^2}}$
9	Uniform thin spherical shell	Inside θ Outside $\frac{GM}{r^2}$	Inside $-\frac{GM}{a}$ Outside $-\frac{GM}{r}$

10	Uniform solid sphere	Inside $\frac{GMr}{a^3}$ Outside $\frac{GM}{r^2}$	Inside $-GMr^2/a^3$ Outside $-\frac{GM}{2a^2}\left(3a^2-r^2\right)$ Here, a is the radius and r is the location of point mass.
11	Gravitational potential	Note: It is a scalar; SI unit is J/kg.	
12		$\vec{E}=-\left[\frac{\partial y}{\partial x}\hat{i}+\frac{\partial V}{\partial y}\hat{j}+\frac{\partial V}{\partial z}\hat{k}\right]$ Note: It is partial derivative $dV=-\vec{E}.\overrightarrow{dr}$.	
13	Gravitational potential energy	$U=-\frac{Gm_1m_2}{r}$ System of particle $\left(m_1\,m_2\,m_3\,m_4\right)$ $U=-G\left[\frac{m_4m_3}{r_{43}}+\frac{m_4m_2}{r_{42}}+\frac{m_4m_1}{r_{41}}+\frac{m_3m_2}{r_{32}}+\frac{m_3m_1}{r_{31}}+\frac{m_2m_1}{r_{21}}\right]$ They are $\frac{4(4-1)}{2}=6$ Pairs	
14	For an n particle system, no. of pairs would be	$\frac{n(n-1)}{2}$ Pairs	
15	Binding Energy  	$E=\frac{GMm}{R}$ It is due to this energy particle is bound to earth.	
16	Escape Velocity	$v_e=\sqrt{2gR}$	
17	Motion of Satellites  	Orbital Speed $v_o=\sqrt{\frac{GM}{r}}$ Time period: $T=\frac{2\pi r}{v_o}=2\pi\sqrt{\frac{r^3}{GM}}$ Energy of satellite: $U=-\frac{GMm}{r};K=\frac{GMm}{2r}$ U is The potential energy Total Energy "E" K is The kinetic energy $E=K+U=-\frac{GMm}{2r}=-K$	

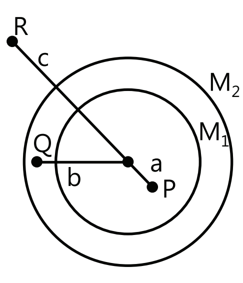
18	Kepler's Laws 	1^{st} Law:- Law of elliptical orbits 2^{nd} Law:- Law of conservation of angular momentum 3^{rd} Law:- Harmonic law $\left(T^2 \propto r^3\right)$ $v_1 r_1 = v_2 r_2$ $v_1 = a(1+e)\ \ r_2 = a(1-e)$ $V_{min} = V_1 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$ $V_{max} = V_2 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$

Solved Examples

JEE Main/Boards

Example 1: Two concentric shells of mass M_1 and M_2 are as shown. Calculate the gravitational force on m due to M_1 at points P, Q and R.

Sol: For a particle of mass m, lying at a distance r from the center of the spherical shell of mass M, the gravitational force of attraction is $\left(\frac{GMm}{r^2}\right)$. If the particle is lying inside the spherical shell then the force of gravitation on it is zero.



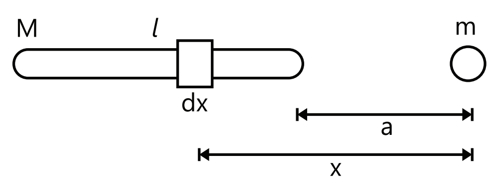
At P, $F = 0$ At Q, $F = \frac{GM_1 m}{b^2}$

At R, $F = \frac{G(M_1 + M_2)m}{c^2}$

Example 2: Find the potential energy of gravitational interaction of a point mass m and a thin uniform rod of mass M and length l, if they are located along a straight line at a distance a from each other.

Sol: The gravitational potential energy is given by $U = \frac{Gm_1 m_2}{r}$ where m_1 and m_2 are point masses. Consider the gravitational potential energy of interaction between the point mass m and an infinitesimal element of the rod of mass dm. The total potential energy will be the summation of energy of interaction of all the small elements.

Consider small element dx of the rod whose mass $dm = \frac{M}{l}dx$



$$\Rightarrow dU = -\frac{Gm\left(\frac{M}{l}dx\right)}{x}$$

$$\Rightarrow U = \int dU = -\frac{GmM}{l}\int_a^{a+l}\frac{dx}{x} = -\frac{GmM}{l}\left[\ln x\right]_a^{a+l}$$

$$\Rightarrow U = -\frac{GmM}{l}\log_e\left(\frac{a+l}{a}\right)$$